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BY

SAMUEL NEWTH, M.A., D.D.,

FELLOW OF UNIVERSITY COLLEGE, LONDON;

PRINCIPAL OF NEW COLLEGE, LONDON.

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PREFACE.

In the present edition this work has been carefully revised throughout, and with special reference to recent changes in the regulations of the University of London. Several new sections and additional examples have been introduced.

At the same time the author has stedfastly kept in view his original design; namely, that of supplying a manual, by the aid of which a Student may intelligently prosecute the study of Natural Philosophy in the earlier stages of his Mathematical course.

The Work is divided into two parts. The First Part embraces the more elementary portions of Mechanics, and can be read by a student who has advanced as far as the earlier propositions of the Sixth Book of Euclid, and has an elementary knowledge of Plane Trigonometry. In the Second Part some of the subjects are carried forward to the point at which a knowledge of the Differential

Calculus becomes necessary. For reading this a moderate knowledge of Algebraical Geometry and of Spherical Trigonometry is requisite.

The changes thus made in the Work will, it is hoped, increase its utility, as well for the purposes of general Education, as for those who are preparing for University Examinations.

SAMUEL NEWTH.

NEW COLLEGE, LONDON,
October, 1879.

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Part I.

STATICS.

CHAPTER I.

DEFINITIONS AND AXIOMS.

1. Whatever is capable of producing motion in a body, or any *change* in the motion of a body, is termed *force*.

In other words, force is the name we employ to express that unknown cause which, under any circumstances, can produce a change in the state, whether of rest or motion, of any material body.

Whatever causes a change in the motion of a body must be regarded as of like nature with that which produces motion, and hence the same term (force) is applied to both, even although there are some forces which, while they are able to change the motion of a body, can never produce it. Such, for example, are friction and resistances of all kinds. Forces of this nature can, it is evident, never act alone, for some other force must be present in order to produce the motion which they change; and hence, if but one force act upon a body, it must be one capable of *producing* motion.

2. If one force only act upon a body, motion must necessarily follow; but when two or more forces act upon the same body,

their united effect *may* be such that no motion ensues. Whenever this is the case, the forces are said to be *in equilibrium.*

3. That branch of mechanics which investigates the relations which exist between forces in equilibrium is termed Statics; and that which investigates the effects of forces not in equilibrium, but producing motion, is termed Dynamics.

4. Whenever motion is prevented by muscular effort, as for instance, when a weight is held in the hand and so prevented from falling to the ground, or when an elastic cord is stretched and its rebound prevented, a sensation is produced which we call *pressure.* But, just as the same word is used in several other cases both for the sensation and its cause—*e.g.* sound, smell, taste —so is the term pressure applied also to the force whose resulting motion has been thus prevented. In this sense, then, pressure is force considered as the cause of the sensation which is felt when motion is prevented by muscular effort. It is with this meaning that the term is commonly employed in mechanics, although extended to all forces when the motion they are capable of producing is in any way prevented, whether it be by muscular action or not. Thus, whether a weight be held in the hand or rest upon a table, in either case it is said to exert a pressure, and a pressure of the same amount.

The commonest case of pressure is weight, and this supplies the most convenient standard of reference by which to compare different pressures. By means of weight, other pressures may in most cases be easily measured. Thus, when an elastic cord is held stretched by the hand, the pressure it exerts may be compared with others, by determining what weight will keep the string stretched to the same degree.

Since all questions considered in Statics refer to forces in equilibrium, all statical forces may be denominated pressures, and consequently may be measured by weight.

5. Forces may differ from each other, not only in magnitude,

but also in direction, and hence may be conveniently represented by straight lines; the direction of the lines representing the direction of the forces, and the lengths of the lines, each being measured by the same scale, the magnitudes of the forces.

Two forces will be called *concurrent*, if they both act towards or from the same point. In the case of parallel forces, and of forces acting along the same straight line, this point is supposed to lie at an infinite distance.

6. Force, when transmitted by means of a cord, is sometimes spoken of as the *tension* in the cord.

7. The single force, which represents the combined effect of several forces, is termed their *resultant;* relatively to the resultant, these several forces are termed *components* or *component forces. Composition of forces* takes place when two or more forces are replaced by a single force equivalent to them, that is, when the resultant is substituted for its components. *Resolution of forces* takes place when a single force is replaced by two or more forces equivalent to it, that is, when the components are substituted for their resultant.

8. A *rigid* body is one the relative position of whose particles is supposed to be invariable.

9. AXIOMS. i. If two non-concurrent forces acting in the same straight line are in equilibrium, they are equal.

ii. If two non-concurrent forces acting in the same straight line are equal, they are in equilibrium.

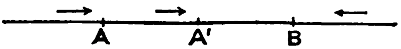
iii. The resultant of two forces acting in the same straight line is their sum when the forces are concurrent, and their difference when non-concurrent.

iv. The resultant of two concurrent forces acting along different lines falls between them.

v. The effect of any forces acting upon a rigid body is unaltered by the introduction or the removal of any number of forces that are mutually in equilibrium.

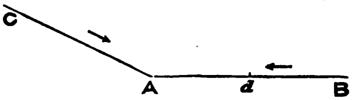
10. Transmissibility of Force. *A force acting upon a rigid body may be supposed to act at any point, within the body, in the line of its direction.*

Thus, if a force P acted upon a rigid body at A, and along the line AB in the direction of the arrow, we may suppose P to be applied at any other point A′ in the line AB within the body. For at A′ and B introduce two non-concurrent forces, each of the same magnitude as P. The forces at A and B are (Ax. ii.) in equilibrium, and may, therefore, by Ax. v., be removed. There will then remain only a force P at A′ of the same magnitude as the original force, and acting in the same direction.



11. *If two forces be in equilibrium their directions must be exactly opposite.*

For, if possible, let two forces acting upon any body along the lines CA, BA, be in equilibrium; since there is equilibrium, there will not less be equilibrium, if any point be supposed fixed. Let d in the line AB be such a point, the force acting along AB will be met by the reaction of the fixed point d. The force CA will then remain alone, and will turn the body round the fixed point d, and therefore there will not be equilibrium. Consequently, if two forces be in equilibrium, their directions must be exactly opposed, and hence, by Axiom i., they are also equal in magnitude.



12. If any number of forces be in equilibrium, any one force must be equal in magnitude and opposite in direction to the resultant of the remaining forces. For if, instead of the remaining forces, we substitute their resultant, we shall have but two forces; namely, this resultant and the force in question, and these are in equilibrium, and therefore are equal and opposite.

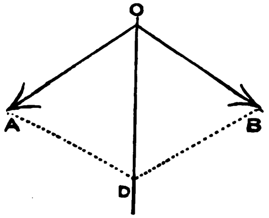
CHAPTER II.

ON THE COMPOSITION AND RESOLUTION OF FORCES.

13. **Parallelogram of Forces.** *If two concurrent forces acting upon a point are represented in magnitude and direction by the two sides of a parallelogram, then will their resultant also be represented in magnitude and direction by the diagonal drawn through the given point.*

First. If two equal forces act upon a point, the *direction* of the resultant is that of the diagonal.

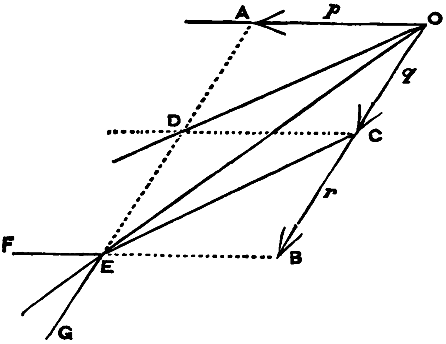
Let AO, BO represent the magnitude and direction of two equal forces acting upon the point O, the diagonal OD must be in the direction of the resultant: for since OADB is an equilateral parallelogram, the diagonal OD bisects the angle AOB; and since the forces acting upon O are equal, there can be no reason why the resultant should pass nearer to the one than to the other; it must therefore pass at an equal distance from both: consequently it must pass along OD.



Secondly. If the direction of the resultant be that of the diagonal, in the case of forces whose magnitudes are p and q, and of forces whose magnitudes are p and r, it is so also in the case of forces whose magnitudes are p and $q+r$.

Let AO, BO represent two forces of the magnitudes p and $q+r$ respectively, acting upon the point O, and suppose them to be kept in equilibrium by some unknown force acting upon O. Let $CO = q$, then $BC = r$.

Complete the parallelograms AC and DB. Draw the diagonals OD, OE, and CE. For the forces p and q in AO and CO substitute their resultant, which, by the hypothesis, is a force acting along OD. Suppose this force to act at D (Art. 10), then resolving it, we have a force p acting along the line DC, and a force q acting along the line DE. Let these two forces act at C and E respectively.



Again, for the forces p in CD and r in CB, substitute their resultant, which, by hypothesis, is a force acting along CE. Let this force act at E, and resolve it. We shall have then at E a force p acting in EF, and two forces q and r acting in EG. We have consequently, without disturbing the equilibrium, removed the original forces p and $q + r$ from O to E. The resultant, therefore, of the given forces must pass through E. It must also pass through O, and therefore must act along the diagonal OE.

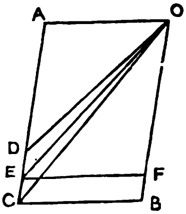
Thirdly. The direction of the resultant is that of the diagonal, in the case of any two *commensurable forces.*

This has already been shown to be true of equal forces, or of the forces p and p; and therefore, by the proposition just proved, (making q and r both equal to p) of forces p and $2p$, therefore also of p and $3p$, and so of p and mp.

Again, the proposition being true of forces mp and p, and of forces mp and p, is true of forces mp and $2p$, therefore also of mp and $3p$, and so of mp and np, m and n being any whole numbers.

Fourthly. The proposition is also true of incommensurable forces.

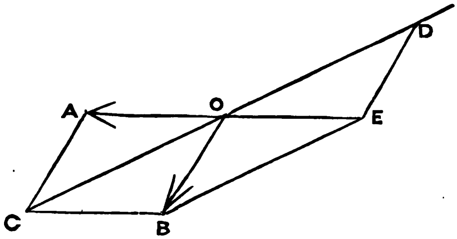
Let AO, BO represent any two incommensurable forces acting upon the point O. If their resultant do not act along the diagonal CO, suppose it to act along some other line, DO. Take any sub-multiple of the line AO that is less than DC. Mark off successive portions equal to this along the line AC, beginning at A. Some one division will ultimately fall between D and C. Let E be this point. Draw EF parallel to AO. Then AO and FO represent commensurable forces. Combining them, we have a force acting along EO. Instead, then, of the forces AO and BO, we have a force in EO and a force in BO represented in magnitude by BF. But, by hypothesis, the resultant of the forces AO and BO is a force in DO; therefore the resultant of a force in EO and a force in BO is a force in DO, which falls without them, which is impossible. (Axiom iv.) Therefore the resultant of AO and BO cannot act along DO; and in a similar manner it may be shown, that it cannot act along any other line above or below the diagonal CO; consequently it must act along the diagonal.



Fifthly. The diagonal represents the resultant in magnitude.

Let OA, OB represent two forces acting upon O. Complete the parallelogram AOBC. Draw the diagonal OC and produce it through O. Make OD equal to OC. Through D draw a line parallel to OB, cutting AO produced in E. Join BE, then ODEB is a parallelogram.

The resultant of OA and OB acts along OC. The forces OA and OB will consequently be kept at rest by a force in OD equal to this resultant. Suppose such a force to act in



OD, then the point O will be kept at rest; but if three forces keep a point at rest, any one must be equal and opposite to the resultant of the other two. Therefore OA must be equal and opposite to the resultant of OB and the force in OD: consequently EO must be this resultant.

But the resultant of any two forces falls along the diagonal of the parallelogram whose sides represent these two forces in magnitude and direction. Therefore OD must represent the magnitude of the force in OD, otherwise the resultant of it and BO would not fall along OE. But the force in OD was taken equal to the resultant of the forces in AO and BO, and OD = OC = the diagonal of the parallelogram AB. Therefore the resultant of any two forces is represented in magnitude, as well as in direction, by the diagonal of the parallelogram, whose sides represent the components.

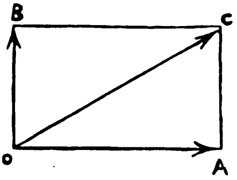
14. It follows from the preceding, that whenever the directions and magnitudes of two forces acting upon a point are known, the direction and magnitudes of their resultant can always be found by a geometrical construction. We have simply to draw from the given point two lines in the given directions, of such lengths that they shall represent the magnitudes of the two forces upon any scale whatever, and then, completing the parallelogram, the direction of the resultant is that of the diagonal drawn through the given point, and its magnitude is represented by the length of this diagonal on the same scale as the components.

It also follows that the magnitude of the resultant of two forces acting upon a point can be determined by calculation, whenever the length of the diagonal of a parallelogram can be so determined. In certain simple cases, this can be done without the aid of trigonometry.

The following are some of these.

Ex. 1. To find the magnitude of the resultant when two forces, P and Q, act upon a point at angle of 90°.

Let OA and OB represent the forces P and Q. Completing the parallelograms, the diagonal OC represents the resultant.



Since the angle BOA is a right angle, the angle OAC is a right angle; therefore, (*Euclid* i. 47,)

$$OC^2 = OA^2 + AC^2.$$

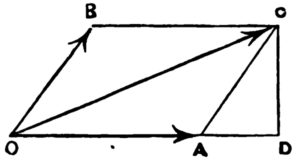
Hence, if R denote the resultant of P and Q,

$$R^2 = P^2 + Q^2,$$

or, $$R = \sqrt{(P^2 + Q^2)}.$$

Ex. 2. To find the magnitude of the resultant when two forces, P and Q, act concurrently upon a point at an angle of 60°.

Let OA and OB represent the forces P and Q. Completing the parallelogram, the diagonal OC represents the resultant or R.



Draw CD at right angles to OA produced, then, (*Euclid* ii. 12,)

$$OC^2 = OA^2 + AC^2 + 2.OA.AD.$$

But since the angle BOA is 60°, the angle CAD is 60°, and the triangle CAD is half an equilateral triangle; therefore,

$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}Q.$$

Hence, $$R^2 = P^2 + Q^2 + PQ,$$

or, $$R = \sqrt{(P^2 + Q^2 + PQ)}.$$

Ex. 3. To find the magnitude of the resultant when two forces, P and Q, act concurrently upon a point at an angle of 45°.

As in the last example, let OA and OB represent the given forces; then, as before,

$$R^2 = P^2 + Q^2 + 2P.AD.$$

But since the angles BOA is 45°, the angle CAD is 45°; therefore, also, ACD is 45°. Whence $DC = AD$; consequently,

$$2AD^2 = AC^2 = Q^2;$$

$$\therefore \quad AD = \tfrac{1}{2}Q\sqrt{2}.$$

Hence, $$R^2 = P^2 + Q^2 + PQ\sqrt{2},$$

or, $$R = \sqrt{(P^2 + Q^2 + PQ\sqrt{2})}.$$

15. *If two forces,* P *and* Q, *act concurrently upon a point at any angle* θ, *and if* R *be their resultant, then* $R^2 = P^2 + Q^2 + 2PQ \cos \theta$.

For let OA and OB in the last figure represent the forces P and Q, then OC represents R.

By trigonometry, $OC^2 = OA^2 + AC^2 - 2.OA.AC \cos OAC$;

$$\therefore \quad R^2 = P^2 + Q^2 - 2PQ \cos OAC;$$

but OAC is the supplement of θ; therefore, $\cos OAC = -\cos \theta$;

$$\therefore \quad R^2 = P^2 + Q^2 + 2PQ \cos \theta.$$

The result just obtained determines the magnitude of R; to determine its direction, we have,

$$\sin COA : \sin OAC :: AC : OC;$$

therefore, if $\hat{RP}$ denote the angle between R and P,

$$\sin \hat{RP} : \sin \theta :: Q : R;$$

$$\therefore \quad \sin \hat{RP} = \frac{Q \sin \theta}{R}.$$

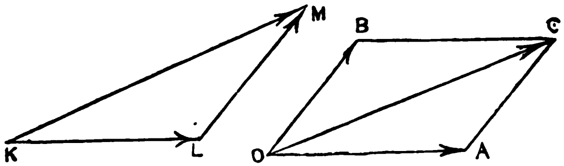
In like manner, $$\sin \hat{RQ} = \frac{P \sin \theta}{R}.$$

16. Since the components may be substituted for their resultant, their effects being equivalent, it follows from the parallelogram of forces, that for any force we may substitute two others, whose magnitudes and directions are represented by the sides of any parallelogram, which has the line representing the given force for its diagonal. And since an infinite number of parallelograms can be drawn, having a given line for their diagonal, any force can be resolved into two others in an infinite number of ways.

17. *If three forces act upon a point, and their resultant be required, find the resultant of any two of them; the composition of this resultant, with the third force, will give the resultant of the three given forces.* In like manner, the resultant of any number of forces acting upon a point may be found. Another, and sometimes more convenient, method is given in Article 24.

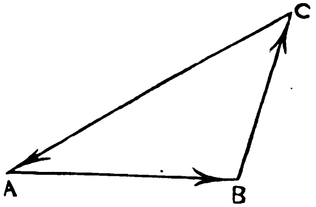
18. *Def.* The sides of any rectilinear figure are said to be taken *in order*, when taken as they would be traversed by a point moving continuously around the figure in either direction, that is to say, either as the hands of a watch revolve, or in the contrary direction. Thus, if ABCD be any quadrilateral, its sides taken in order are either AB, BC, CD, DA, or AD, DC, CB, BA. It will be sometimes convenient to describe these relatively to each other, as in direct and reverse order respectively.

19. *If two sides of a triangle, taken in order, represent in magnitude and direction two forces acting upon a point, then shall the third side, taken in reverse order, represent the resultant in magnitude and direction.*



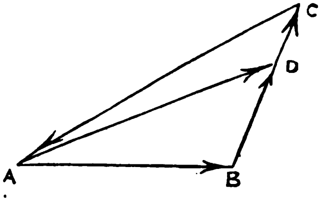
Let the sides KL, LM, of the triangle KLM, representing in magnitude and direction the forces P and Q acting at any point O, then will KM represent in magnitude and direction the resultant of P and Q. Through O draw OA equal and parallel to KL, and pointing in the same way; also OB equal and parallel to LM, and pointing in the same way: then OA and OB represent P and Q respectively. Completing the parallelogram, the diagonal OC represents the resultant of P and Q. But since the lines OA, AC, are parallel respectively to KL, LM, and point in the same way, the angle OAC is equal to the angle KLM. Then in the triangles KLM, OAC, the sides KL, LM, and the angle KLM are equal severally to the sides OA, AC, and the angle OAC; therefore KM is equal to OC. It can also be easily shown that KM is parallel to OC; therefore the line KM represents OC, the resultant of P and Q, both in magnitude and direction.

20. Triangle of Forces. *If three forces, represented in magnitude and direction by the sides of a triangle taken in order, act upon a point, they will be in equilibrium;* and, conversely, *if three forces acting upon a point, and in equilibrium, be represented in direction by the sides of a triangle taken in order, they will also be represented in magnitude by the sides of that triangle.*



First. Let three forces, represented in magnitude and direction by the sides of the triangle ABC, taken in order, that is to say, by AB, BC, and CA, act upon any point, these forces will be in equilibrium.

For, by the preceding article, the resultant of the forces represented by AB and BC is a force represented by AC; that is, a force equal and opposite to the force represented by CA. The forces are therefore in equilibrium.



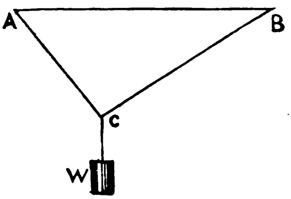
Secondly. Let any three forces, P, Q, and R, acting upon a point, be in equilibrium, and let the sides of the triangle ABC, taken in order, represent the direction of these forces; viz., AB the direction of P; BC, that of Q; and CA, that of R. Then will these sides, AB, BC, CA, represent severally the magnitudes of P, Q, and R.

Let AB represent P on any scale, then must BC represent Q on the same scale. For, if possible, let some other length BD represent Q; then, since AB, BD, two sides of a triangle taken in order, represent in magnitude and direction two forces acting upon a point, the third side AD, by Art. 19, represents their resultant. But, by hypothesis, there is equilibrium; and therefore a force represented in direction by AD is balanced by a force represented in direction by CA, which is impossible. Therefore, BD cannot represent the magnitude of Q. In like manner it may be shown, that no other length than BC can represent Q. Then, since AB represents P, and BC represents Q, it follows that CA represents R.

21. The triangle whose sides are severally equally inclined to the sides of the triangle ABC, is equiangular to ABC, and consequently its sides are in the same ratio. Hence, *if three forces be in equilibrium, and any triangle be drawn whose sides are severally either parallel or equally inclined to their directions, the forces are to one another as the sides of the triangle.*

The following are examples of the application of the triangle of forces to the solution of mechanical problems.

Ex. 1. A weight W is sustained by two cords of given lengths CA and CB, fastened to two points A and B, lying in the same horizontal line; to determine the tensions in the cords, when the cords are at right angles.



Let a and b be the lengths of the cords CA and CB, then since ACB is a right angle, AB $=\sqrt{(a^2+b^2)}$.

The point C is at rest under the action of three forces; viz., W acting vertically, and the tensions in CA and CB. In the triangle ACB,

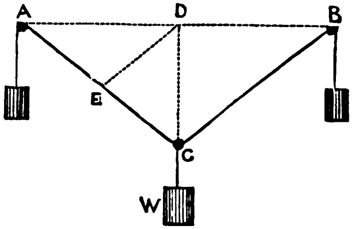
AB is perpendicular to W,
BC ,, the tension in CA,
CA ,, the tension in CB;

$$\therefore \quad \text{tension in CA} : \text{W} :: \text{BC} : \text{AB}$$
$$:: b : \sqrt{(a^2+b^2)};$$
$$\therefore \quad \text{tension in CA} = \frac{\text{W}b}{\sqrt{(a^2+b^2)}}.$$

Similarly,

$$\text{tension in CB} = \frac{\text{W}a}{\sqrt{(a^2+b^2)}}.$$

Ex. 2. Two cords bearing equal weights, P, P, pass over pulleys placed at A and B in the same horizontal line, and are joined to a third weight W at C; what must this weight be that C may rest at a given distance below AB?



Let the vertical line through C meet the line AB in D, thence since there is no reason why C should be nearer to A than to B, AB will be bisected in D. Let AB = $2a$, and CD = b.

Through D draw DE parallel to BC, then AC is bisected in E.

In the triangle CDE,

DC is parallel to W,
CE ,, the tension in CA,
ED ,, the tension in CB.

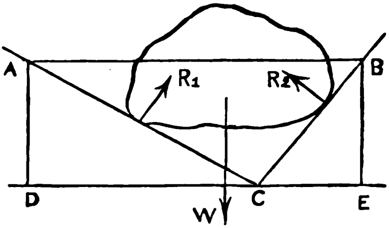
Therefore by the triangle of forces,

$$\text{W} : \text{tension in CA} :: \text{DC} : \text{CE}.$$

But the tension in CA is equal to P, and EC $= \frac{1}{2}$AC $= \frac{1}{2}\sqrt{(a^2+b^2)}$;

$$\therefore \quad \text{W} : \text{P} :: b : \tfrac{1}{2}\sqrt{(a^2+b^2)};$$
$$\therefore \quad \text{W} = \frac{2\text{P}b}{\sqrt{(a^2+b^2)}}.$$

Ex. 3. A body whose weight is 168 lbs. rests upon two smooth inclined planes; one of the planes rises 3 in 5, and the pressure upon it is 160 lbs.: to determine the inclination of the other plane, and also the pressure upon it.



Let AC, BC, be the two planes; the plane AC rising 3 in 5.

Take AC = 5, and through A draw AD perpendicular to CD, the horizontal line through C; then will AD = 3; and, therefore, DC $=\sqrt{(25-9)}=4$. Through A draw B horizontally, and through B draw BE perpendicular to CE; then BE = AD = 3; and if x stand for BC, CE $=\sqrt{(x^2-9)}$.

Three forces are in equilibrium; viz., W the weight of the body, and the resistances R_1, R_2. In the triangle ABC,

AB is perpendicular to W,
BC ,, R_2,
CA ,, R_1.

Therefore, by the triangle of forces,

$$R_1 : W :: CA : AB.$$

But, by hyp., $R_1=160$, $W=168$, $CA=5$, and $AB=DC+CE=4+\sqrt{(x^2-9)}$. Hence,

$$160 : 168 :: 5 : 4+\sqrt{(x^2-9)};$$

$$\therefore \quad \sqrt{(x^2-9)}=\frac{5}{4};$$

$$\therefore \quad x=3\tfrac{1}{4},$$

or the plane BC rises 3 in $3\frac{1}{4}$, or 12 in 13.

Also, $R_2 : R_1 :: BC : CA$;

$\therefore$ $R_2 : 160 :: 3\frac{1}{4} : 5$,

or, $R_2 = 104$ lbs.

22. *If three forces acting upon a point be in equilibrium, they are severally as the sine of the angle contained between the other two.*

Let P, Q, and R, be three forces acting upon a point, and in equilibrium. It has been shown, in Art. 20, that if the sides of the triangle ABC, taken in order, be severally parallel to the directions of these forces, then,

$$P : Q : R :: AB : BC : CA;$$

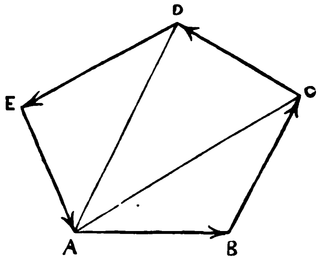
but, by trigonometry,

$$AB : BC : CA :: \sin BCA : \sin CAB : \sin ABC.$$

But if $\widehat{PQ}$ denote the angle between P and Q, $\widehat{PQ} = 180° - ABC$; and therefore $\sin ABC = \sin \widehat{PQ}$. Similarly, $\sin CAB = \sin \widehat{RP}$, and $\sin BCA = \sin \widehat{QR}$. Therefore,

$$P : Q : R :: \sin \widehat{QR} : \sin \widehat{RP} : \sin \widehat{PQ}.$$

23. Polygon of Forces. *If any number of forces acting upon a point be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium.*



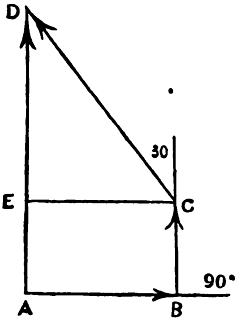
Let forces represented by the sides of the polygon ABCDE, taken in order, act upon any point, they will be in equilibrium; for, by Art. 19, the resultant of the forces represented by AB and BC will be represented by AC. In like manner, the resultant of the forces represented by AC and CD will be represented by AD; or AD represents the resultant of the forces represented by AB, BC, and CD. Substituting this resultant for its components, we have remaining three forces, represented by AD, DE, EA, three sides of a triangle taken in order; and, therefore, by the triangle of forces, they are in equilibrium.

24. If any number of forces be in equilibrium, a force equal and opposite to any one will be the resultant of the remaining forces. Hence any side of a polygon, taken in reverse order, will represent the magnitude and direction of the resultant of any number of forces acting upon a point, when these forces are represented in magnitude and direction by the remaining sides of the polygon taken in direct order.

Thus, if AB, BC, CD, and DE in the last figure represent in magnitude and direction four forces acting upon any point, the remaining side AE (not EA) will represent the magnitude and direction of their resultant.

Hence, to find the magnitude and direction of any number of forces acting concurrently upon any point, draw a line parallel to one of the forces, and representing it in magnitude. Through that extremity of this line which points in the same way as the force, draw a second line parallel to the next force, pointing in the same way, and representing it in magnitude. Through the extremity of this second line, draw a third line parallel to the third force, pointing in the same way, and also representing it in magnitude. Proceed in this way until lines have been drawn representing all the forces. The straight line which completes the polygon, taken in reverse order, will represent the resultant in magnitude and direction.

Example. Three forces, P, Q, and 2P, act upon a point; the angle between the first and second is 90°, and the angle between the second and third is 30°, required the magnitude of the resultant.



Through any point A, draw AB equal and parallel to P. Through B, draw BC equal and parallel to Q; and through C, draw CD equal and parallel to the third force 2P. Join A, D, then AD represents the resultant.

To find the length of AD: draw CE parallel to AB, and DE perpendicular to CE; join E, A.

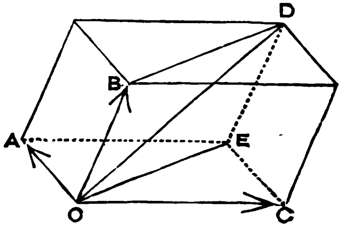
By construction, ABC is 90^c, and BCD is 150°; therefore the angle DCE is 60°, and the triangle DCE is half an equilateral triangle. Hence, CE is one-half of CD, that is, equals AB or P, and $DE = P\sqrt{3}$. Then, since CE is equal and parallel to AB, the figure EB is a parallelogram; and, consequently, the angle AEC is a right angle. But the angle DEC is also a right angle, therefore the lines AE, ED are in the same straight line. Therefore,

$$\begin{aligned} AD &= AE + ED, \\ &= Q + P\sqrt{3}. \end{aligned}$$

25. *If three forces, acting upon a point, be represented in magnitude and direction by the three edges of a parallelopiped, their resultant will be represented in magnitude and direction by the diagonal.*

Let OA, OB, OC, the three edges of the parallelopiped BE, represent in magnitude and direction three forces acting upon O.

Since AC is a parallelogram, the diagonal OE represents the resultant of the forces OA and OC. Compound this resultant OE with the third force OB, then since BE is a parallelogram, their resultant is represented by OD; and therefore OD represents the resultant of the three forces OA, OB, OC.



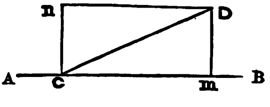
26. Lemma. *No force can produce any effect in a direction perpendicular to its own.*

For there can be no reason why it should cause the body to move towards one side rather than towards the other; and, consequently, it will not cause the body to move to either side.

27. *To determine the total effect of a force in any given direction.*

Resolve the force into two other forces, one perpendicular to, and the other acting along, the given direction. It follows from the preceding Lemma, that the last-mentioned force is the thing sought.

Thus, let CD represent a given force, and it be required to determine its total effect in the direction AB. Draw the rectangle *nm*. Then the force CD may be resolved into two forces represented by C*n* and C*m*. The force C*n* produces no effect in the direction AB, and therefore C*m* is the force required.



Let P be the given force, and let CD $= r$, and C$m = x$, then the required force : P : : x : r,

or, the required force $= \mathrm{P}\frac{x}{r}$.

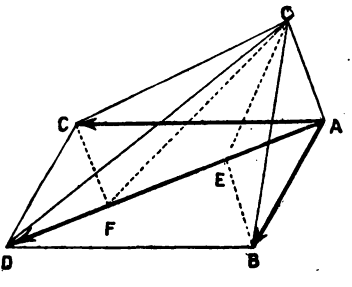
Let the angle DC$m = \theta$, then the total effect of a force P, in a direction inclined to its own, at an angle θ, is P$\cos\theta$.

28. *Def.* The product of a force, and the perpendicular distance of its direction from any given point, is termed the *moment* of the force about that point. When two forces tend to produce rotation in the same direction about any point, their moments about that point are said to be of the *same sign;* but when they tend to produce rotation in contrary directions, they are said to be of *contrary signs.*

29. *The moment of the resultant of any two converging forces about a point in their plane is equal to the algebraic sum of the moments of the two forces about that point.*

First, let the point lie without the angle contained between the two forces, then whatever parallelogram be drawn with two of its sides representing the forces, this point will be outside the parallelogram.

Let AB, AC, sides of the parallelogram ABDC, represent the two forces, and let O be the given point. The diagonal AD represents the resultant. Draw OA, OB, OC, and OD. Through B and C draw BE and CF parallel to OA. Draw OF and OE. Then it may easily be shown that the triangles ABE, CDF are equal in all respects, and hence that AE = DF.



Because BE is parallel to OA,

triangle OAB = triangle OAE.

And because CF is parallel to OA,

triangle OAC = triangle OAF.

But because DF is equal to AE,

triangle ODF = triangle OAE,

∴ triangle OAB = triangle ODF.

But, triangle OAD = triangle ODF + triangle OAF,

∴ triangle OAD = triangle OAB + triangle OAC.

Now, every triangle equals half the product of its base and height; and hence if P and Q be the two forces, and R their resultant, and if p q, and r be the perpendiculars drawn from O to P, Q, and R respectively, the triangle OAD = $\frac{1}{2}\mathrm{R}r$, the triangle OAB = $\frac{1}{2}\mathrm{P}p$, and the triangle OAC = $\frac{1}{2}\mathrm{Q}q$.

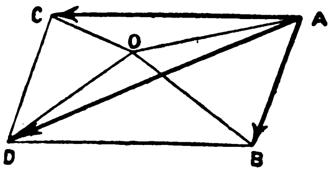
Hence, $\frac{1}{2}\mathrm{R}r = \frac{1}{2}\mathrm{P}p + \frac{1}{2}\mathrm{Q}q$,

or, $\mathrm{R}r = \mathrm{P}p + \mathrm{Q}q$.

In this case the moments of the two forces about O are of the same sign, and the moment of the resultant is equal to the sum of the moments of the two forces.

Secondly, let the point be within the angle contained by the two forces. Then the parallelogram whose sides represent the forces may be so drawn as that the point shall be within the parallelogram.

Let AB, AC represent the two forces, and let O be the given point. The diagonal AD represents the resultant. Draw OA, OB, OC, OD.



The two triangles OAB, OCD are together equal to half the parallelo-gram.

The triangles OAD, OAC, OCD are together equal to half the parallelogram. Therefore,

triangle OAD + triangle OAC = triangle OAB,

or, triangle OAD = triangle OAB − triangle OAC.

Hence if, as before, P and Q be the two forces, R their resultant, and p, q, r the perpendiculars from O to P, Q, and R respectively,

$$\tfrac{1}{2}Rr = \tfrac{1}{2}Pp - \tfrac{1}{2}Qq,$$

or,

$$Rr = Pp - Qq.$$

Here the moment of P and Q are of contrary signs, and consequently $Pp - Qq$ is their *algebraic* sum. Hence in all cases the moment of the resultant is equal to the algebraic sum of the moments of the two forces.

30. It follows from the preceding that if any number of forces act upon a point, the moment of their resultant about any point whatever is equal to the algebraic sum of the moments of the several forces about that point.

Let P_1, P_2, P_3 be the forces, p_1, p_2, p_3 the distances of the forces from any point O. Then if R_1 be the resultant of P_1 and P_2, and r_1 its distance from O. Then by Art. 29,

$$R_1r_1 = P_1p + P_2p_2,$$

and if R be the resultant of R_1 and P_3, and r its distance from O: then since R and P_3 are converging forces, it follows also that

$$Rr = R_1r_1 + P_3p.$$

And hence,

$$Rr = P_1p_1 + P_2p_2 + P_3p_3.$$

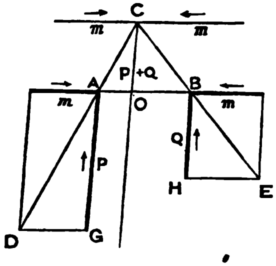
In like manner the proposition may be established for any number of forces acting upon a point.

Further on a similar proposition will be established in the case of any forces acting in a plane, whatever their directions and positions.

31. If in the preceding the point of reference be in the direction of the resultant itself, then, since in this case $r = 0$, the left-hand side of the equation vanishes, or the right-hand side is equal to zero. Hence, *If any number of forces act upon a point, the algebraic sum of their moments about any point in the direction of their resultant is zero.*

In the case of a body moving about a fixed point, it is clear that no motion can result from the action of any force whose line of direction passes through that point. The effect of such a force is simply to press the body against the fixed point. Hence, also, no motion can result from the combined action of several forces if their resultant passes through the fixed point; and the resultant will so pass if the algebraic sum of the moments of the forces about the point be equal to zero. This is true universally, but at present is established only so far as relates to forces whose lines of direction meet in one and the same point.

32. *To determine the magnitude and position of the resultant of two concurrent parallel forces* P *and* Q.



In the directions of P and Q take any points A and B, and introduce at A and B two equal and opposite forces (m, m). The resultant of m and P is AD, and the resultant of m and Q is BE. Remove the points of application of these two resultants from A and B to C, and resolve them into their original components. We shall then have two equal and opposite forces m, m, which will destroy each other, and two concurrent forces P and Q acting along CO. The resultant therefore is a force whose magnitude is $P + Q$ acting along CO.

Because the triangles DGA, AOC are similar, $m : P :: AO : OC$;

$$\therefore \quad P \times AO = m \times OC.$$

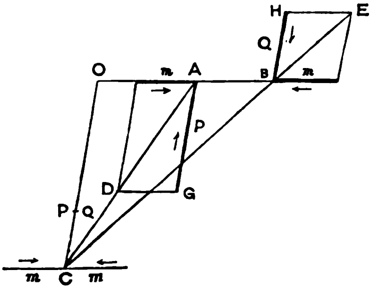
$$\text{Similarly,} \quad Q \times BO = m \times OC,$$

$$\therefore \quad P \times AO = Q \times BO.$$

33. *To find the resultant of two non-concurrent parallel forces* P *and* Q.

As in the former case, take any two points A and B in the directions of P and Q, and introduce two equal and opposite forces (m, m).

Compound these with P and Q. Produce the directions of the resultants till they meet in C. Remove the point of application of these resultants from A and B to C, and resolve them into their original components. We shall then have two equal and opposite forces (*m*, *m*) acting in a line parallel to AB, which will destroy each other, and along CO parallel to the directions of P and Q, two non-concurrent forces equal severally to P and Q. The resultant then will be a force $P-Q$ acting along CO.



Because the triangles DGA, AOC are similar, $m : P :: AO : OC$;

$$\therefore \qquad P \times AO = m \times OC.$$

$$\text{Similarly,} \qquad Q \times BO = m \times OC,$$

$$\therefore \qquad P \times AO = Q \times BO.$$

34. From these investigations we see, in the case of two parallel forces ;—That the resultant is always parallel to the components ;—That when the components are concurrent, the resultant is concurrent with them, and equal to their sum ; but when the components are non-concurrent, the resultant is equal to their difference, and concurrent with the greater ;—That if any line whatever (for the points A and B were taken quite arbitrarily) be drawn across the directions of the components, it will be cut by the resultant at a point such that the one force multiplied into its distance from the resultant measured along this line, equals the other force multiplied into its distance.

35. In Art. 33, if the forces P and Q had been equal, the construction would have been impossible. The lines AD, BE would have been parallel, and consequently the determination of the point C would have become an impossibility. The two forces will

therefore have no resultant. Such a combination of forces is termed a *couple.*

36. *The resultant of any number of concurrent parallel forces equals the sum of the components.*

Let P_1, P_2, P_3, P_4, be the forces. Let R_1 be the resultant of P_1 and P_2, then

$$R_1 = P_1 + P_2.$$

Let R_2 be the resultant of R_1 and P_3; then

$$\begin{aligned} R_2 &= R_1 + P_3, \\ &= P_1 + P_2 + P_3. \end{aligned}$$

The resultant of R_2 and P_4 will be the resultant of the given forces; let this be R, then

$$\begin{aligned} R &= R_2 + P_4, \\ &= P_1 + P_2 + P_3 + P_4; \end{aligned}$$

and similarly of any other number of parallel forces.

37. Hence the resultant of any number of parallel forces equals their algebraic sum.

Let P_1, P_2, &c., and Q_1, Q_2, &c., be any number of parallel forces, of which P_1, P_2, &c., act in one direction, and Q_1, Q_2, &c., in the opposite direction. Let X be the resultant of P_1, P_2, &c., and Y the resultant of Q_1, Q_2, &c., and let X be greater than Y. Then, since P_1, P_2, &c., are concurrent forces,

$$X = P_1 + P_2 + \&c.$$

For a similar reason,

$$Y = Q_1 + Q_2 + \&c.$$

If R be the resultant of the given forces, R must be the resultant of X and Y; but these are non-concurrent forces, therefore

$$R = X - Y,$$

$$\therefore \quad R = P_1 + P_2 + \&c. - Q_1 - Q_2 - \&c.$$

38. *Two concurrent parallel forces,* P *and* Q, *act at a distance* a *from each other, to find the distance of the resultant from either force.*

Let x be the distance of the resultant from P; then, since the resultant passes between P and Q, its distance from Q will be $a - x$. Therefore

$$Px = Q\,(a - x),$$

or,
$$(P + Q)\,x = Qa,$$

$$\therefore \quad x = \frac{Qa}{P + Q} = \frac{Qa}{R},$$

or the distance of the resultant from one of the components equals the distance between the components multiplied by the other component, and divided by the resultant.

39. *Two non-concurrent forces,* P and Q, *act at a distance* a *from each other, to find the distance of the resultant from either force.*

Let P be the greater force, and R their resultant, then R equals $P - Q$, and passes outside P. Let x be the distance of P from the resultant, then $a + x$ is the distance of Q. Therefore

$$Px = Q\,(a + x),$$

or,
$$(P - Q)\,x = Qa,$$

$$\therefore \quad x = \frac{Qa}{P - Q} = \frac{Qa}{R},$$

or, as before, the distance of the resultant from either component equals the distance between the components multiplied by the other component, and divided by the resultant.

40. *To resolve a given force* P *into two parallel forces, acting at given distances on each side of it.*

Let a and b be the given distances, and let X and Y be the forces required, then P will be the resultant of X and Y; consequently $X + Y = P$, and $aX = bY$;

$$\therefore \quad bX + bY = Pb,$$

or,
$$bX + aX = Pb,$$

$$\therefore \quad X = P\frac{b}{a + b}, \text{ and } \therefore\ Y = P\frac{a}{a + b}.$$

Hence, to find either component, multiply the given force by the distance of the other component, and divide by the distance between the two components.

41. *To resolve a given force* P *into two parallel forces, acting at given distances on the same side of it.*

Let a and b be the given distances, a being the greater, and X and Y the required forces. From Art. 34, it appears that X and Y will be non-concurrent, and that the force nearer to P, namely, Y, will act concurrently with P. Also, Y – X equals P, and aX equals bY;

$$\therefore \qquad bY - bX = Pb,$$

$$\text{or,} \qquad aX - bX = Pb,$$

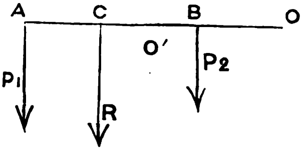
$$\therefore \qquad X = P\frac{b}{a-b}, \text{ and } \therefore Y = P\frac{a}{a-b};$$

or, as before, to find either component, multiply the given force by the distance of the other component, and divide by the distance between the two components.

42. *The algebraic sum of the moments of any two parallel forces about any point in their plane is equal to the moment of the resultant about that point.*

Let P_1 and P_2 be the two forces, and let R be their resultant: let p_1, p_2, and r be their distances respectively from the given point.

First. Let the forces be concurrent, and let the point (O) lie without them. From O draw a line perpendicular to the directions of the forces, meeting them in A and B and the resultant in C. Then Art. 32,



$$P_1.AC = P_2.BC.$$

But $AC = AO - CO = p_1 - r$ and $BC = CO - BO = r - p_2$,

$$\therefore \qquad P_1(p_1 - r) = P_2(r - p_2),$$

$$\text{or,} \qquad P_1p_1 + P_2p_2 = (P_1 + P_2)r = Rr.$$

Secondly. Let the forces be concurrent, and let the point (O′) be between them. Then, as before,

$$P_1 . AC = P_2 . BC.$$

But $AC = AO' - CO' = p_1 - r$ and $BC = BO' + CO' = p_2 + r$.

$$\therefore \qquad P_1 (p_1 - r) = P_2 (p_2 + r),$$

$$\text{or,} \qquad P_1 p_1 - P_2 p_2 = (P_1 + P_2) r = Rr.$$

As the two forces P_1 and P_2 tend to produce rotation about O′ in contrary directions, their moments are of contrary signs, and hence $P_1 p_1 - P_2 p_2$ is the algebraic sum of their moments.

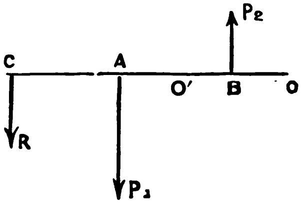
Thirdly. Let the forces be non-concurrent, and let the point (O) be without them. As before,

$$P_1 . AC = P_2 . BC.$$

But $AC = CO - AO = r - p_1$; and
$BC = CO - BO = r - p_2$,

$$\therefore \qquad P_1 (r - p_1) = P_2 (r - p_2),$$

$$\text{or,} \qquad P_1 p_1 - P_2 p_2 = (P_1 - P_2) r = Rr.$$



Here again the two forces tend to produce rotation about O in contrary directions, and $P_1 p_1 - P_2 p_2$ is the algebraic sum of their moments.

Fourthly. Let the forces be non-concurrent, and let the point (O′) be between them. As before,

$$P_1 . AC = P_2 . BC.$$

But $AC = CO' - AO' = r - p_1$ and $BC = BO' + CO' = p_2 + r$.

$$\therefore \qquad P_1 (r - p_1) = P_2 (p_2 + r),$$

$$\text{or,} \qquad P_1 p_1 + P_2 p_2 = (P_1 - P_2) r = Rr.$$

Here the two forces produce rotation about O′ in the same direction; their moments therefore are of the same sign.

43. *If any number of parallel forces act in the same plane, the algebraic sum of their moments about any point in the plane is equal to the moment of the resultant about that point.*

Let the forces be P_1, P_2, P_3, P_4, &c., and their perpendicular distances from the given point p_1, p_2, p_3, p_4, &c., respectively. Let R_1 be the resultant of P_1 and P_2, and its distance from the given point r_1, then, by Article 42,

$$R_1r_1 = P_1p_1 + P_2p_2.$$

Let R_2 be the resultant of R_1 and P_3, and its distance from the given point r_2;

$$\therefore \qquad R_2r_2 = R_1r_1 + P_3p_3,$$
$$= P_1p_1 + P_2p_2 + P_3p_3,$$

and similarly for the rest; and hence if R be the resultant of all the forces, and its distance from the given point be r, then

$$Rr = P_1p_1 + P_2p_2 + P_3p_3 + P_4p_4 + \&c.$$

COR. *If the given point be in the direction of the resultant* $r = o$, *and the left-hand member of the equation vanishes. Hence if any number of parallel forces act in the same plane, the algebraic sum of their moments about any point in their resultant will be equal to zero.*

And conversely, *if the algebraic sum of the moments of any number of parallel forces acting in the same plane, and not in equilibrium, about any point in the plane, be equal to zero, that point is in the direction of the resultant.*

For if possible, let the sum of the moments about a point not in the resultant be equal to zero; then, if x be the distance of this point from the resultant, $p_1 - x$, $p_2 - x$, &c., will be the distances of the forces from this point; and therefore, by hypothesis,

$$P_1(p_1 - x) + P_2(p_2 - x) + \&c. = o.$$

But, by the preceding,

$$P_1p_1 + P_2p_2 + \&c. = o;$$

therefore, subtracting,

$$(P_1 + P_2 + \&c.)x = o;$$

or, $$Rx = o.$$

But, by hypothesis, the forces are not in equilibrium, that is, R does not vanish; and therefore $x = o$, or the assumed point must be in the resultant.

44. *If any number of forces parallel or convergent act in the same plane, the algebraic sum of their moments about any point in their plane is equal to the moment of the resultant about that point.*

Let P_1, P_2, P_3, &c., be the forces, p_1, p_2, p_3, &c., the distances of their lines of direction from the point of reference. Then if R_1 be the resultant of P_1 and P_2, and r_1 its distance from the point of reference, then whether P_1 and P_2 be convergent or be parallel, it has been seen, Articles 29, 32, 33, that

$$P_1p_1 + P_2p_2 = R_1r_1.$$

Again, let R_2 be the resultant of R_1 and P_3, and r_2 its distance from the point of reference; then whether R_1 and P_3 be convergent or be parallel,

$$R_1r_1 + P_3p_3 = R_2r_2$$

and $\therefore$ $$P_1p_1 + P_2p_2 + P_3p_3 = R_2r_2.$$

Hence, whatever the number of forces, if R be the resultant and r its distance from the point of reference,

$$P_1p_1 + P_2p_2 + P_3p_3 + \&c. = Rr.$$

Cor. i. If the point of reference be in the line of direction of the resultant, then $r = 0$ and the right-hand side of this equation vanishes; and hence, *If any number of forces parallel or convergent act in the same plane, the algebraic sum of their moments about any point in their resultant is equal to zero.*

Cor. ii. If the forces be in equilibrium, then $R = 0$ and the right-hand of the equation vanishes, wherever the point of reference may be; and hence, *If any number of forces acting in a plane be in equilibrium, the algebraic sum of their moments about any point whatever in their plane is equal to zero.*

45. Equations of Equilibrium. If each of any number of forces acting on a plane be resolved into two others, one parallel to a given direction, and the other perpendicular to it, the entire set of forces is reduced to two sets of *parallel* forces, one set being parallel to the given direction, and the other set perpendicular to it.

If the given forces be in equilibrium, then the algebraic sum of each of these sets is equal to zero. For the forces in each set being parallel forces, their resultant is their algebraic sum; and if the resultant of either set be other than zero, then the given forces would have a resultant other than zero; that is, would not be in equilibrium.

Hence, if P_1, P_2, &c., be any forces in equilibrium, acting at angles α_1, α_2, &c., with a given line, then the components parallel to this line are $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, &c., and the components perpendicular to this line are $P_1 \sin \alpha_1$, $P_2 \sin \alpha_2$, &c.; and hence,

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0$$
$$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \&c. = 0.$$

And these equations, combined with that given in the preceding section,

$$P_1 p_1 + P_2 p_2 + \&c. = 0,$$

are the equations which exhibit the relations subsisting between any set of forces acting in a plane and in equilibrium.

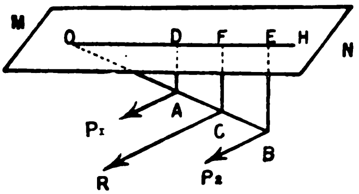
46. If all the given forces act upon a point, then the third equation is necessarily satisfied; since if we take that point as the point of reference, the moment of each force about that is zero. Hence, in the case of forces acting upon a point and in the same plane, the relations subsisting between them when in equilibrium are expressed by the two equations,

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0$$
$$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \&c. = 0.$$

47. *Def.* The product of a force, and the perpendicular distance of its point of application from any given plane, is termed the moment of the force with respect to the plane.

48. *The algebraic sum of the moments of two parallel forces with respect to any plane whatever is equal to the moment of their resultant with respect to that plane.*

Let P_1 and P_2 be two parallel forces acting at the points A and B. Let R be their resultant, meeting the line AB in C. Let MN be any plane, and let perpendiculars to the plane from A, B, and C meet the plane in D, E, and F, and let the lengths of these perpendiculars be h_1, h_2, and h.



Then, if AB be parallel to the plane, h_1, h_2, and h are all equal, and consequently since $R = P_1 + P_2$, Art. 37,

$$Rh = P_1h_1 + P_2h_2.$$

If AB be not parallel to the plane, let AB, or AB produced, meet it in O. Let a plane drawn through AB perpendicular to MN cut MN in the line OH, then the lines AD, BE, and CF will lie in this plane. Let the angle BOH be θ. Then, by Art. 42,

$$R.CO = P_1.AO + P_2.BO,$$

$$\therefore \quad R.CO \sin\theta = P_1.AO \sin\theta + P_2.BO \sin\theta.$$

But CO $\sin\theta = h$, AO $\sin\theta = h_1$, and BO $\sin\theta = h_2$; therefore,

$$Rh = P_1h_1 + P_2h_2.$$

49. *The algebraic sum of the moments of any number of parallel forces with respect to any plane whatever is equal to the moment of the resultant with respect to that plane.*

Let the forces be P_1, P_2, P_3, &c., acting at points whose perpendicular distances from the plane are h_1, h_2, h_3, &c.

Let the resultant of P_1 and P_2 be R_1; let it act at a distance z_1 from the given plane; then, by Article 48,

$$R_1z_1 = P_1h_1 + P_2h_2.$$

Let the resultant of R_1 and P_3 be R_2; and let it act at a distance z_2 from the given plane; then,

$$\begin{aligned} R_2z_2 &= R_1z_1 + P_3h_3, \\ &= P_1h_1 + P_2h_2 + P_3h_3; \end{aligned}$$

and in like manner for the rest. Hence, if R be the resultant of all the forces, and h the distance of its point of application from the given plane,

$$Rh = P_1h_1 + P_2h_2 + P_3h_3 + P_4h_4 + \&c.$$

COR. If the plane pass through the point of application of the resultant, $h = 0$; hence, in such a case, the algebraic sum of the moments of the forces equals zero.

50. It has been seen in Art. 37 that the resultant of any number of parallel forces, whether acting in one plane or not, is equal to the algebraic sum of the components.

The *position* of the resultant may be determined thus:

Take any two planes parallel to the direction of the forces, then, by the proposition given in the preceding section, the moment of the resultant in respect to each of these planes is equal to the algebraic sum of the moments of the components. It is generally most convenient to take these planes at right angles to each other.

Hence, if P_1, P_2, &c., be the forces, h_1, h_2, &c., the distances of any points in their lines of direction from one of the planes of reference, and k_1, k_2, &c., their distances from the other plane, and if R be the resultant, h and k the distances of any point in its line of direction from the two planes, then the equations for the determination of R, h, and k are

$$\begin{aligned} R &= P_1 + P_2 + \&c., \\ Rh &= P_1h_1 + P_2h_2 + \&c., \\ Rk &= P_1k_1 + P_2k_2 + \&c. \end{aligned}$$

COR. If the forces be in equilibrium, $R = 0$, hence the equations of equilibrium for any number of parallel forces are

$$\begin{aligned} P_1 + P_2 + \&c. &= 0, \\ P_1h_1 + P_2h_2 + \&c. &= 0, \\ P_1k_1 + P_2k_2 + \&c. &= 0. \end{aligned}$$

EXAMPLES.

1. Two forces of 7 and 33 lbs. act upon a point at an angle of 60°, required the resultant. Ans. 37 lbs.

2. Two forces of 5 and 21 lbs. act upon a point at an angle of 120°, required their resultant. Ans. 19 lbs.

3. The resultant of two equal forces acting upon a point at an angle of 90° is 100 lbs., required the value of each component. Ans. 70·7107 lbs.

4. The resultant of two equal forces acting upon a point at an angle of 135° is 10 lbs., required the value of each component. Ans. 13·0659 lbs.

5. The resultant of two forces acting upon a point at an angle of 150° is 19 lbs., one of the components is 11 lbs., find the other component. Ans. $16\sqrt{3}$ lbs.

6. Two forces of 9 lbs. and 56 lbs. acting upon a point are balanced by a force of 61 lbs., show that the angle between the two forces is 60°.

7. Two forces of 7 lbs. and 40 lbs. acting upon a point are balanced by a force of 37 lbs., show that the angle between the two forces is 120°.

8. Two forces acting upon a point at an angle of 120° have a resultant whose magnitude is $P\sqrt{3}$, and which makes an angle of 30° with one of the forces, find the magnitude of the two forces.

The two forces are P and 2P.

9. Three equal forces acting upon a point are in equilibrium, show that the angle between any two of them is 120°.

10. If two forces P and 3P act upon a point at an angle of 120°, show that the magnitude of the resultant is not affected by doubling the smaller force.

11. Five forces each equal to P act upon a point in such a way that the first is at right angles with the second; the third is at right angles with the resultant of the first and second; the fourth, at right angles to the resultant of the first, second, and third; and the fifth, at right angles with the resultant of the other four; required the magnitude of the resultant of the five forces.

Ans. $P\sqrt{5}$.

12. If six equal forces act upon a rigid body along the sides of a regular hexagon, in such a way that the forces act concurrently at each angle, show that the forces will be in equilibrium.

13. Three forces, whose magnitudes are severally 16, 63, and 156 lbs., act upon a point at right angles to each other, show by the parallelopiped of forces that the resultant is 169 lbs.

14. Three forces, of 9, 15, and 28 lbs. act upon a point concurrently; the angle between the first and second is 60°, and the third is at right angles with each of the other two: required the magnitude of the resultant. Ans. 35 lbs.

15. A smooth ring, sustaining a weight W, is carried by a cord fastened at two points, A and B, in the same horizontal line; required the tension in the cord.

Let $2b$ be the length of the cord, and $2a$ the distance AB, then the tension in the cord will be equal to

$$\frac{Wb}{2\sqrt{(b^2-a^2)}}.$$

16. Required the horizontal force necessary to draw, on a level road, a carriage wheel over an obstacle of given height.

Let W be the weight of the carriage, a the radius of the wheel, and c the height of the obstacle, then the required force will be equal to

$$\frac{W.\sqrt{(2ac-c^2)}}{a-c.}$$

17. Two strings, fastened to two pegs, A and B, in a vertical wall, of which A is the higher, are joined at the point C to a weight W; the strings are of such lengths that BC is horizontal, and the angle BCA is 135°: find the tensions in the strings.

The tension in AC is equal to $W\sqrt{2}$, and the tension in BC is equal to W.

18. Four equal forces act concurrently upon a point, and in the same plane, the angle between the first and second, that between the second and third, and that between the third and fourth, are each equal to 72°: find the magnitude of the resultant.

The resultant is equal to one of the components.

19. A ball, whose weight is W, slides along a smooth rod, inclined at an angle of 30° with the vertical line; what force applied in the direction of the rod will keep the ball at rest.

Ans. $\frac{1}{2}W\sqrt{3}$.

20. A heavy ball, which slides along a smooth rod, inclined at an angle of 45° with the vertical line, is kept at rest by a force P acting along the rod; required the weight of the ball.

Ans. $P\sqrt{2}$.

21. Two forces of 65 and 84 lbs. act upon a point at an angle of 70°, required the magnitude of the resultant.

Ans. 122·54 lbs.

22. Two forces of 20 and 64 lbs. act upon a point; their resultant is 76 lbs., what is the angle between the two forces?

Ans. 60°.

23. Two forces of 3 and 5 lbs. act upon a point; their resultant is 7 lbs., what is the angle between the resultant and the larger of the two forces?

Ans. $\sin^{-1}\left(\frac{3\sqrt{3}}{14}\right)$

24. One of two components is double the other, and their resultant is equal to half their sum; required the angle between them.

Ans. $\cos^{-1}\left(-\frac{11}{16}\right)$

25. The resultant of two forces acting upon a point is equal to one half of one of them, and is at right angles with the other; required the angle between the two forces. Ans. 150°.

26. If X, Y, and Z be three forces acting upon a point, and the angle between X and Y be equal to α, and the angle between Y and Z be equal to β, and if R be the resultant, then

$$R^2 = X^2 + Y^2 + Z^2 + 2XY \cos \alpha + 2YZ \cos \beta + 2ZX \cos (\alpha + \beta).$$

27. A weight W, to which two cords are attached, rests upon a smooth horizontal table; one cord passes over a pulley placed at the edge of the table, and bears a weight Q; the other cord passes over a pulley placed above the table, and bears a weight P; required the inclination of the latter cord to the table?

Ans. $\cos^{-1}\left(\frac{Q}{P}\right)$.

28. One end of a cord is fastened to a fixed point A, the other is carried over a fixed pulley at B, and fastened to a weight P; what weight must be attached to the cord at a point C, midway between AB, that the part AC may rest horizontally?

Let AB be inclined to the horizon at an angle α, then the required weight will be equal to

$$\frac{2P \sin \alpha}{\sqrt{(5 - 4 \cos \alpha)}}.$$

29. In the preceding, show that the tension in AC is equal to

$$\frac{P(1 - 2 \cos \alpha)}{\sqrt{(5 - 4 \cos \alpha)}}.$$

30. A particle, whose weight is W, rests against the circumference of a circular plate, whose plane is vertical; a cord attached to the particle passes over a pulley placed vertically above the highest point of the circle, at a distance from the circle equal to the radius; required the position in which the particle will rest, and the pressure on the circle, when a weight P hangs from the cord.

In the position of equilibrium, the inclination of the cord to the vertical line is equal to

$$\cos^{-1}\left(\frac{3W^2+4P^2}{8WP}\right).$$

and the pressure on the circle is equal to

$$\tfrac{1}{2}W.$$

31. A bar AB, 5 feet 4 inches in length, rests horizontally upon two vertical props A and B; a weight of 320 lbs. is suspended from the bar at a distance of 27 inches from A: required the pressures upon the props.

Ans. 185 lbs. on A, and 135 lbs. on B.

32. Two weights P and Q, of which P is the greater, are suspended from the extremities of a straight bar; where must a third weight W be attached that the bar may balance about its middle point?

Let $2a$ be the length of the bar, then the weight W must be attached at a distance from the middle point, on the side of Q, equal to

$$\frac{(P-Q)\,a}{W}.$$

33. Three concurrent parallel forces, of 15, 20, and 24 lbs., act in the same plane upon a rigid body, and at distances of 12, 14, and 30 inches respectively to the right of a certain point in their plane: required the distance of their resultant from the same point.

Ans. 20 inches.

34. If a weight W rests upon a triangular table ABC, at a point O, show that the pressure on the leg A is to W as the triangle BOC is to the triangle ABC.

35. If the magnitude of four concurrent parallel forces be 8, 12, 15, and 25 respectively, and the distances of their points of application from a given plane be 5, 10, 8, and 6, what is the distance of the point of application of their resultant from the same plane? Ans. $7\frac{1}{6}$.

36. If the distances of the points of application of n equal concurrent parallel forces from a plane passing through the point of application of the resultant form a descending arithmetical series, of which the first term is a, and common difference b, show that

$$\frac{a}{b} = \frac{n-1}{2}.$$

37. Resolve a given force R into three parallel components, P_1, P_2, P_3, acting at given points. Let z be the distance of the point of application of R from any line in the plane of the points, and z_1, z_2, z_3 the distances of the given points from the same line, and let h, h_1, h_2, h_3 be the respective distances of the same points from any other line; then shall

$$P_1 = R\frac{(z\,h_2 - h\,z_2) + (z_2h_3 - h_2z_3) + (z_3h\, - h_3z\,)}{(z_1h_2 - h_1z_2) + (z_2h_3 - h_2z_3) + (z_3h_1 - h_3z_1)}.$$

38. Show that the resolution of any force into more than three parallel components, acting at given points in one plane, is an indeterminate problem.

39. A bar is formed of unequal lengths of two substances, each of uniform thickness and density, the one weighing m oz. per inch, and the other n oz. per inch; how far from the middle point must a fulcrum be placed that the bar may balance about it by its own weight?

Let a be the length in inches of the one part, and b that of the other, then the distance required is

$$\frac{ab}{2}\cdot\frac{m \backsim n}{ma+nb}.$$

40. A series of n equal weights is suspended from a weightless bar, in such a way that the distance between the first and second is a, that between the second and third $2a$, that between the third and fourth $3a$, and so on, the successive intervals forming an arithmetic series; required the distance of the resultant from the first weight.

Assuming that the sum of n terms of the series $1+3+6+10+\&c.+\frac{n.(n+1)}{2}$ is $\frac{n.(n+1)(n+2)}{6}$, show that the distance required is equal to $\frac{1}{6}(n^2-1)a$.

41. Weights P, 2P, 3P, &c....nP are suspended at distances a, $2a$, $3a$, &c....na from one extremity of a weightless rod; required the distance of the resultant from that extremity.

Ans. $\frac{1}{3}(2n+1)a$.

42. A straight weightless bar ABCD rests horizontally upon two pegs at B and C; required the pressures upon the pegs, when weights P and Q are suspended from A and D.

Let $AB=a$, $BC=c$, and $CD=b$; then the pressure at B is equal to $\frac{P(a+c)-Qb}{c}$,

and the pressure at C is equal to $\frac{Q(b+c)-Pa}{c}$.

43. A weightless bar ACB, bent so as to form a right angle at C, is moveable about a fixed axis at C; required the position of equilibrium, when weights P and Q are suspended from A and B.

Let $AC=a$, and $BC=b$; then, if θ be the inclination of AC to the horizontal line,

$$\tan\theta=\frac{Pa}{Qb}.$$

44. Weights P and Q, of which P is the greater, are suspended by cords from A and B, opposite points in the rim of a weightless hemispherical bowl, resting upon a horizontal table; required the

position of equilibrium, both weights hanging outside of the bowl.

The inclination of AB to the vertical line is equal to

$$\operatorname{Sin}^{-1}\left(\frac{Q}{P}\right).$$

45. Two equal bars, AB and BC, of uniform thickness an density, rigidly connected at B, when suspended from A, rest wit BC horizontal; required the angle ABC. Ans. $\cos^{-1}(\frac{1}{3})$.

46. Two equal bars, AB and BC, of uniform thickness and density, rigidly connected at B, are suspended from A; required the inclination of the bar AB to the vertical line through A, in the position of equilibrium.

Let the angle ABC be equal to α; then the inclination required is equal to

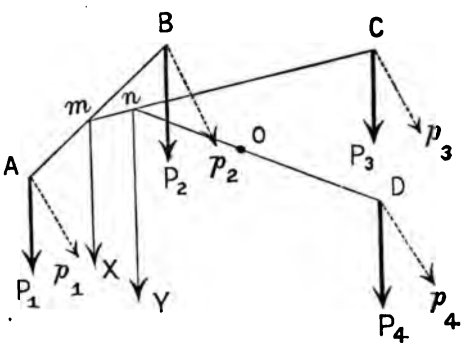
$$\cot^{-1}\left(\frac{3-\cos\alpha}{\sin\alpha}\right).$$

CHAPTER III.

ON THE CENTRE OF GRAVITY.

51. *If any number of parallel forces act at given points in a rigid body, there is a point through which the resultant will always pass, whatever the position in which the body may be placed.* This point is termed the centre of the parallel forces.

Let P_1, P_2, P_3, P_4 be parallel forces acting at A, B, C, D, points in a rigid body. Join the points A, B, and in AB take m, so that $Am : mB :: P_2 : P_1$, then, by Art. 32, the resultant of P_1 and P_2 passes through m. Let X be this resultant, and join the points m, C; in mC take n, so that $mn : n\mathrm{C} :: P_3 : X$, then as before the resultant of X and P_3, that is of P_1, P_2, and P_3 passes through n. Let Y be this resultant, and join n, D : in nD take o, so that $no : o\mathrm{D} :: P_4 : Y$, then the resultant of Y and P_4, that is of P_1, P_2, P_3 and P_4, passes through o.



If now we suppose the body to be moved into a new position, or what would be the same thing, if we suppose the direction of the several forces to be changed, and to assume the position represented in the figure by p_1, p_2, p_3, and p_4, then, since the magnitudes of the forces are unchanged, the resultant of p_1 and p_2 will as before pass through m, that of p_1, p_2, and p_3, through n, and that of p_1, p_2, p_3, and p_4, through o. The same will take place in any other position of the body, and hence in all positions of the body the resultant of the several forces will pass through o.

52. *Gravity* is the force which attracts every particle of matter towards the centre of the earth. The *weight* of a body is the total force with which that body is drawn towards the earth's centre; it is therefore the same as the resultant of the forces which, in consequence of the existence of gravity, act upon its several particles. These forces may, without sensible error, be regarded as parallel forces. The centre of these parallel forces is called the *centre of gravity*.

The centre of gravity of a body is, then, the point through which will pass, in every position of the body, the resultant of all the forces which, in consequence of gravity, act upon its particles.

53. It follows from the definition, that if the centre of gravity of a body be fixed, the body will rest in any position. For in every position the resultant of all the forces arising from gravity passes through the fixed point, and being met by the resistance of that point, equilibrium is preserved.

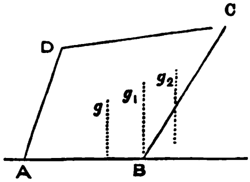
54. Since the resultant may be always substituted in place of its component forces, it follows that in considering the influence of any weighty body in producing equilibrium, we may substitute a force equal to the weight of the body, and acting at its centre of gravity.

55. *If a body suspended from any point be at rest, the centre of gravity must lie in the vertical line drawn through this point.* For there are two forces in equilibrium; viz., the weight of the body which acts vertically and at the centre of gravity, and the reaction of the fixed point. By Art. 11, these forces must lie in the same straight line, and therefore the vertical line through the centre of gravity must pass through the fixed point.

Hence, if a body be suspended successively from two different points, and the vertical lines passing through those points be drawn, the intersection of these two lines will be the centre of gravity.

56. *A body placed with its base upon a plane surface will stand or fall, according as the vertical line through its centre of gravity falls within or without the base.*

Let ABCD be any body, whose base AB rests upon a plane surface. Let the centre of gravity be at g, so that the vertical line through g falls within AB. The weight of the body acting in this line is met by the resistance of the plane, and, consequently, will not cause the body to turn about either A or B.



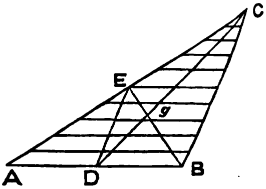
But if the centre of gravity be at g_2, so that the vertical line drawn through the centre of gravity falls without the base, the weight of the body, acting in this line, is not met by the re-action of the plane. There cannot, therefore, be equilibrium, but the body will turn over on to the side BC.

If the centre of gravity be at g_1, so that the vertical line drawn through it pass through one of the extremities of the base, the body will rest; for the weight acting in the vertical line is met by the resistance of the plane: a very slight disturbance will, however, cause the body to fall over.

57. The centre of any perfectly symmetrical figure must also be the centre of gravity; for there can be no reason why the centre of gravity should fall upon one side of that point more than upon another. Hence, the centre of gravity of a line is its point of bisection; of a parallelogram, the intersection of its diagonals; of a parallelopiped, the intersection of its diagonals; of a circle and sphere, their centres.

58. *To find the centre of gravity of a triangle.* Let ABC be the

given triangle. Bisect AB in D and AC in E. Draw CD, BE. If we suppose the triangle to be made up of an infinite number of lines parallel to AB, these lines will be bisected by CD. Consequently the centre of gravity of each of these lines will be in the line CD, and therefore the centre of gravity of the triangle will lie in the line CD. For similar reasons the centre of gravity must lie in the line BE. Therefore g, the intersection of these two lines, must be the centre of gravity required.



Join the points D, E. DE will be parallel to CB, and the triangle ADE similar to the triangle ABC. Hence,

$$DE : BC :: AD : AB;$$
$$\therefore \quad DE = \tfrac{1}{2}BC.$$

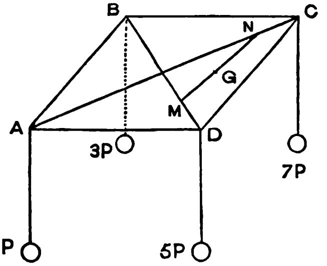
But, by similarity of triangles, $gD : gC :: DE : BC$;

$$\therefore \quad gD = \tfrac{1}{2}gC,$$
$$\therefore \quad gD = \tfrac{1}{3}DC.$$

Hence, the centre of gravity of a triangle lies in the line joining any vertex with the bisection of the opposite side, at a distance from the side of one-third of this line.

59. *To find the centre of gravity of four weights,* P, 3P, 7P, *and* 5P, *suspended from the angular points of a square weightless plate.*

Let ABCD be the square plate, and let the weights P, 3P, 7P, and 5P be suspended severally from the points A, B, C, D. Draw the diagonals, AC, BD. Take N, so that $CN = \tfrac{1}{8}AC$, and M so that $DM = \tfrac{3}{8}BD$. Then $AN = \tfrac{7}{8}AC$, and $BM = \tfrac{5}{8}BD$.



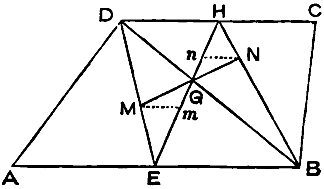
The resultant of P at A, and 7P at C, is a force 8P at N, since $7P \times CN = P \times AN$. And the resultant of 3P at B and 5P at D is also a force 8P at M; since 5

$\times$ DM $= 3$P $\times$ BM. We have, then, in the place of the original forces, two equal forces acting at M and N. Bisect MN at G, then the resultant of these equal forces will pass through G, and G is the centre of gravity required.

60. *To find the centre of gravity of a trapezium.*

Let ABCD be the trapezium, in which the sides AB, CD are parallel.

Bisect AB in E, and DC in H, and join HE. If the trapezium be supposed to be made up of an infinite number of lines parallel to AB, each of these lines will be bisected by EH, and consequently the centre of gravity of the trapezium must be in the line EH.



Join DE, and take EM $= \frac{1}{3}$ED. Similarly join HB, and take HN $= \frac{1}{3}$HB. Then M is the centre of gravity of the triangle ABD, and N that of the triangle BCD. Consequently the centre of gravity of the whole figure must be in the line MN; but it also lies in the line EH; therefore the intersection of these lines, or the point G, is the centre of gravity required.

Let GE be called x, and GH y; and let AB $= a$, and DC $= b$. Draw Mm, Nn parallel to AB or CD. Then, since the triangles EMm, EDH are similar,

$$\therefore \quad \text{M}m : \text{DH} :: \text{EM} : \text{ED};$$

$$\therefore \quad \text{M}m = \tfrac{1}{3}\text{DH} = \tfrac{1}{6}b.$$

$$\text{Similarly,} \quad \text{N}n = \tfrac{1}{3}\text{EB} = \tfrac{1}{6}a.$$

Again, because the triangles MGm, NGn are similar,

$$\text{G}m : \text{G}n :: \text{M}m : \text{N}n;$$

$$\therefore \quad \text{GE} - \text{E}m : \text{G}h - \text{H}n :: \text{M}m : \text{N}n.$$

But Em and Hn are each equal to $\frac{1}{3}$ EH, or $\frac{1}{3}(x+y)$;

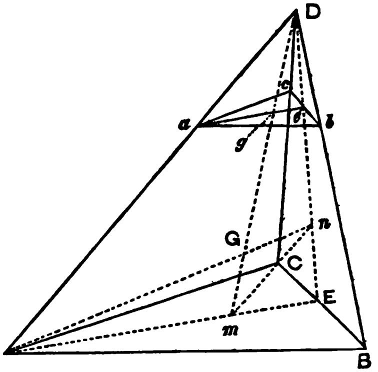
$$\therefore \quad x - \tfrac{1}{3}(x+y) : y - \tfrac{1}{3}(x+y) :: \tfrac{1}{6}b : \tfrac{1}{6}a;$$

$$\therefore \quad 2x - y : 2y - x :: b : a;$$

$$\therefore \quad x : y :: a + 2b : 2a + b.$$

61. *To find the centre of gravity of a triangular pyramid.*

Let ABCD be any triangular pyramid. Bisect BC in E. Draw AE and DE. Take $Em = \frac{1}{3}AE$, and $En = \frac{1}{3}DE$. Join Dm and An. These lines will intersect in a point G, which is the centre of gravity of the pyramid.



The pyramid may be conceived to be made up of an infinite number of planes parallel to ABC. Let *abc* be such a plane. Then, since the parallel planes *abc*, ABC are intersected by a third plane BCD, the intersections *cb*, CB are parallel. And since *cb* is parallel to CB, *cb* is bisected in *e* by the line DE; for, by similarity of triangles,

$$ce : CE :: Dc : DC,$$

and $$cb : CB :: Dc : DC;$$

∴ $$ce : cb :: CE : CB;$$

but $$CE = \frac{1}{2} CB;$$

∴ $$ce = \frac{1}{2} cb.$$

The lines *ae*, AE, are parallel, for they lie in the same plane AED, and in the parallel planes *abc*, ABC. Then, by similarity of triangles

$$eg : Em :: De : DE,$$

and $$ea : EA :: De : DE;$$

∴ $$eg : ea :: Em : EA,$$

but $$Em = \frac{1}{3} EA;$$

∴ $$eg = \frac{1}{3} ea;$$

therefore *g* is the centre of gravity of the triangle *abc*. In like manner, it may be shown that the centre of gravity of every plane parallel to ABC is in the line Dm. The centre of gravity of the whole pyramid must therefore be in the line Dm. For similar reasons it must also be in the line An. Therefore G, the intersection of these two lines, must be the centre of gravity of the pyramid.

Join mn; then, since Am : mE : : Dn : nE, mn is parallel to AD, and therefore

$$mn : \text{AD} :: m\text{E} : \text{AE};$$

$$\text{but} \qquad m\text{E} = \tfrac{1}{3}\,\text{AE};$$

$$\therefore \qquad mn = \tfrac{1}{3}\,\text{AD}.$$

The triangles AGD, mGn are similar; therefore,

$$m\text{G} : \text{GD} :: mn : \text{AD};$$

$$\therefore \qquad m\text{G} = \tfrac{1}{3}\,\text{GD};$$

$$\therefore \qquad m\text{G} = \tfrac{1}{4}\,\text{D}m.$$

Therefore the centre of gravity of the pyramid is one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

62. *To find the centre of gravity of a pyramid whose base is any polygon.*

Divide the polygon into triangles; and if planes be supposed to pass through the sides of these triangles and the vertex, the pyramid will be divided into triangular pyramids.

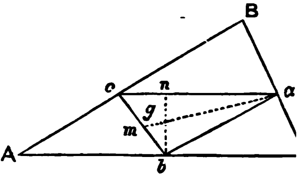
Let a plane be drawn parallel to the base, at a distance equal to one-fourth of the altitude of the pyramid, then (Art. 61) the centre of gravity of each of the triangular pyramids will be in this plane.

Again, join the vertex with the centre of gravity of the base. The centre of gravity of every section parallel to the base will be in this line. Hence, the centre of gravity of the whole pyramid will be in this line.

Therefore the centre of gravity is in the line joining the vertex with the centre of gravity of the base, at a point whose distance from the base is one-fourth of this line.

63. *To find the centre of gravity of the perimeter of a triang*

Let ABC be the given triangle. Join *abc*, the points of bi tion of the three sides. Draw *am*, *bn*, bisecting the angles *cab*, *abc* respectively. The point *g*, the intersection of these two lines, will be the centre of gravity of the perimeter of the triangle ABC. For the three lines, AB, BC, CA, may be conceived to act at the points *c*, with forces represented by their several lengths. The trian *abc*, ABC are similar; and, therefore,



$$AB : BC : CA :: ab : bc : ca.$$

Hence, the force at c : the force at b :: $ab : ca$. But, since bisects the angle bac,

$$bm : mc :: ab : ac;$$

$$\therefore \quad \text{force at } c : \text{force at } b :: bm : mc.$$

Consequently the resultant of the forces at b and c passes thr m, and may be represented in value by $ab + ac$. The resultan this force and the force at a, represented by bc, must pass thr g; for

$$\begin{array}{lrl} & gm : ga & :: mb : ab; \\ \text{but,} & mb : mc & :: ab : ac; \\ \therefore & mb : mb + mc & :: ab : ab + ac; \\ \text{or,} & mb : bc & :: ab : ab + ac; \\ \therefore & mb : ab & :: bc : ab + ac; \\ \text{and hence,} & gm : ga & :: bc : ab + ac. \end{array}$$

The point g is also the centre of the circle inscribed in abc. H the centre of gravity of the perimeter of a triangle is the cent the circle inscribed in the triangle formed by joining the bisect of its sides.

64. If the magnitudes and centres of gravity of the several of a body of uniform density are known, the centre of gra

of the whole body can be found by the proposition given in Art. 49.

For let M be the total magnitude of the body, and M_1, M_2, M_3, &c., the magnitudes of its several parts. But instead of these bodies, we may suppose forces proportional to their weights to act at their respective centres of gravity. Since the density of the body is uniform, these weights will be proportional to the magnitudes. Consequently, if h be the distance of the centre of gravity of the whole body from any plane, and h_1, h_2, h_3, &c. the known distances of the centres of gravity of the several parts from the same plane;

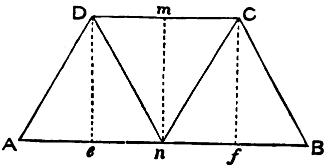
$$Mh = M_1 h_1 + M_2 h_2 + M_3 h_3 + \&c.$$

$$\therefore \quad h = \frac{M_1 h_1 + M_2 h_2 + M_3 h_3 + \&c.}{M}.$$

Ex. 1. To find the centre of gravity of half a regular hexagon.

Let ABCD be half a regular hexagon. Bisect AB in n, and join Dn, Cn. Then the given figure is divided into three equilateral triangles, whose centres of gravity are known.

Bisect DC in m, and join mn. Then the centre of gravity must evidently be in this line, and will consequently be found, if we can determine its distance from AB.



Let D$e = 3c$, and let the area of each of the equilateral triangles be a. Then the area of the whole figure is $3a$. The distance of the centre of gravity of the triangles ADn, BCn, from AB; or, which is the same thing, its distance from a plane, drawn through AB perpendicular to the plane of the paper, is c. The distance of the centre of gravity of the triangle CDn from AB is $2c$. Then if h denote the distance of the centre of gravity of the whole figure from AB,

$$h = \frac{ac + 2ac + ac}{3a} = \frac{4c}{3}.$$

Ex. 2. Let the upper face of a cube be the base of a square pyramid, to find the centre of gravity of the whole figure, the density being uniform.

Let $a =$ the edge of the cube, and $b =$ the height of the pyramid. The centre of gravity must evidently lie in the line joining the vertex of the pyramid with the centre of the base of the cube. Let the base of the cube be the plane from which the distances of the centres of gravity are reckoned.

The content of the cube is a^3, and the distance of its centre of gravity from the base is $\frac{1}{2}a$.

The content of the pyramid is $\frac{1}{3}a^2b$, and the distance of its centre of gravity from the base of the figure is $a + \frac{1}{4}b$. Therefore, if h denote the distance of the centre of gravity of the whole figure from the same plane,

$$h = \frac{a^3(\frac{1}{2}a) + \frac{1}{3}a^2b(a + \frac{1}{4}b)}{a^3 + \frac{1}{3}a^2b},$$

$$= \frac{6a^2 + 4ab + b^2}{12a + 4b}.$$

65. In like manner the centre of gravity of any part can be found, when the centres of gravity of the whole and the remaining parts are known.

Ex. The exterior of a cup is a cylinder, and its interior is a cone; required the depth of the centre of gravity.

Let A be the area of the circle formed by the rim of the cup, a the height of the cup, and b the depth of the vertex of the cone; then the content of the cylinder is Aa, and the depth of its centre of gravity is $\frac{1}{2}a$.

Also, the content of the cone is $\frac{1}{3}Ab$, and the depth of its centre of gravity is $\frac{1}{4}b$.

Therefore the content of the cup is $A(a - \frac{1}{3}b)$; and if x be the depth of its centre of gravity,

$$\frac{Aa^2}{2} = \frac{Ab^2}{12} + A(a - \tfrac{1}{3}b)x;$$

$$\therefore \qquad x = \frac{6a^2 - b^2}{12a - 4b}.$$

EXAMPLES.

1. Four heavy particles, whose weights are 1, 3, 5, and 7 lbs., are placed along a weightless rod, whose length is a, so as to divide its length into three equal parts; required the distance of the centre of gravity from the middle point of the rod. Ans. $\frac{5a}{24}$.

2. Three heavy particles, whose weights are P, Q, and R, are connected at given distances by a weightless rod; required the distance of the centre of gravity from the weight Q.

Let a be the distance between P and Q, and b the distance between Q and R; then the distance required is

$$\frac{Pa \sim Rb}{P+Q+R}.$$

3. Find the distance of the centre of gravity of four-fifths of a regular pentagon from the centre of the inscribed circle.

Let a be the radius of the inscribed circle, then the distance required is $\frac{1}{6}a$.

4. A bar, of uniform thickness and density, is bent so as to form four sides of a regular pentagon; find the distance of the centre of gravity from the centre of the inscribed circle.

Let a be the radius of the inscribed circle, then the distance required is $\frac{1}{4}a$.

5. Find geometrically the centre of gravity of half a regular octagon.

6. Find geometrically the centre of gravity of a bent lever of uniform thickness and density.

7. If from a square ABCD, whose diagonals intersect in O, tl portion AOD be taken away, the distance of the centre of gravi of the remainder from O is equal to $\frac{1}{9}a$ (a being the side of tl square).

8. From a cone, whose height is 8 inches, a similar cone, inches in height, is cut off by a plane parallel to the base; wh is the distance of the centre of gravity of the frustum from tl base? Ans. $1\frac{4}{7}$ inch.

9. The height of a conical shell is 14 inches, and the extern diameter of the base is double the internal diameter, what is tl distance of the centre of gravity from the base?

Ans. $3\frac{3}{4}$ inches.

10. Upon AB, a side of a square ABCD, describe an equilater triangle, and let VN be the line joining its vertex with the bise tion of the side CD; then, if G be the centre of gravity of tl whole figure,

$$GN : VN :: 4-\sqrt{3} : 4+\sqrt{3}.$$

11. If from any pyramid whose height is a, a pyramid who height is b be cut off by a plane parallel to the base, show th the distance of the centre of gravity of the frustum from tl base is equal to

$$\frac{a-b}{4} \cdot \frac{a^2+2ab+3b^2}{a^2+ab+b^2}.$$

12. Let two spheres of the same density, whose radii are a ar b respectively, touch one another externally; the distance of the centre of gravity from the centre of the sphere whose radius a is equal to

$$\frac{b^3}{a^2-ab+b^2}.$$

13. Particles, whose weights are as 3, 4, 5, and 8, are placed the corners of a square weightless plate whose side is 28 inche what is the distance of the centre of gravity from the centre the plate? Ans. $\frac{7}{5}\sqrt{10}$ inch.

CHAPTER IV.

ON THE SIMPLE MACHINES.

66. A *machine* is an instrument, by the agency of which one force can resist or overcome another force not immediately opposite to it in direction.

67. The simplest of such instruments are sometimes termed the *Mechanical Powers*, but are more fittingly named as the *Simple Machines*. These are, the lever, the wheel and axle, toothed wheels, the pulley, the inclined plane, the wedge, and the screw. These differ from each other more in their structure than in the principles of their operation; for when in equilibrium, the wheel and axle, the toothed wheel and the pulley, may be reduced to the lever; and the wedge and the screw are but modifications of the inclined plane.

The two forces which act upon either of these simple machines are, for the sake of distinction, called the *power* and the *weight*, the latter always denoting the force to be resisted or overcome.

68. By the *mechanical advantage* of any machine is meant the ratio of the weight to the power, when in equilibrium; thus, if a power of 2 lbs. sustain a weight of 30 lbs. the mechanical advantage is $30 \div 2$, or 15.

69. THE LEVER. The lever is an inflexible bar, capable of free motion about a fixed axis, called the fulcrum. Unless the contrary be stated, the lever is usually supposed to be without weight.

Levers are of three kinds, according to the relative position of the power, weight, and fulcrum.

When the fulcrum is between the power and the weight, the lever is of the first kind.

When the weight is between the fulcrum and the power, the lever is of the second kind.

When the power is between the fulcrum and the weight, the lever is of the third kind.

The beam of a balance is a lever of the first kind, an oar is a lever of the second kind, and the treadle of a lathe is a lever of the third kind.

Scissors are double levers of the first kind, nut-crackers are double levers of the second kind, and spring shears are double levers of the third kind.

70. *To find the condition of equilibrium in the lever, when the power and the weight act in parallel directions.*

It is evident that there will be equilibrium, if the resultant of the two forces passes through the fulcrum. Let P represent the power, and W the weight; let a be the distance from the fulcrum at which P acts, and b the distance at which W acts. Then, since P and W are by hypothesis parallel forces, their resultant will pass through the fulcrum, if $Pa = Wb$, whether the fulcrum be between the forces, as in levers of the first kind, or without them, as in levers of the second and third kinds. Hence, when the power and the weight are parallel forces, the condition of equilibrium in a lever of any kind is, that the *power* × *its distance from the fulcrum* = *the weight* × *its distance from the fulcrum;* and, therefore, the mechanical advantage

$$= \frac{\text{distance of power from fulcrum}}{\text{distance of weight from fulcrum}}.$$

71. *To find the condition of equilibrium in a lever acted upon by parallel forces, when the weight of the lever itself is regarded.*

Let P denote the power, and W the weight. Let $a =$ the length of the arm at which P acts, and $b =$ the length of the arm at which W acts. Let $w =$ the weight of the lever, and $c =$ the distance of its centre of gravity from the fulcrum.

The weight of the lever may be regarded as a force w acting at the centre of gravity. By Art. 43, there will be equilibrium, if the algebraic sum of the moments about the fulcrum be equal to zero.

First. Let the weight of the lever tend to produce revolution in the same direction as P, then

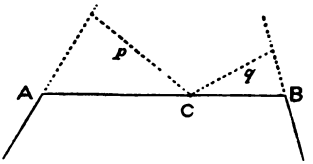
$$Pa + wc - Wb = 0.$$

Secondly. Let the weight of the lever tend to produce revolution in the same direction as W, then

$$Pa - wc - Wb = 0.$$

72. *To determine the condition of equilibrium in a lever, when the power and weight act in any direction whatever.*

Let P and Q be forces acting in any direction at the extremities A and B of any lever, which may be either straight, as in the figure, or bent in any manner. From the fulcrum C let fall perpendiculars upon the direction of the forces, and let their lengths be p and q respectively. Then (Art. 31) the resultant of P and Q passes through C, if $Pp = Qq$, which is, consequently, the condition of equilibrium required.



Cor. If a and b be the arms of any lever, straight or bent, and if P and Q make angles α and β respectively with the arms; then $p = a \sin \alpha$, and $q = b \sin \beta$; consequently, the condition of equilibrium is

$$Pa \sin \alpha = Qb \sin \beta.$$

73. *To determine the pressure upon the fulcrum of a lever.*

The pressure upon the fulcrum is in every case the resultant of the forces acting upon the lever. When these are parallel and act concurrently, the resultant equals their sum; if not concurrent, the resultant equals their difference. Hence, the pressure upon the fulcrum,

in levers of the first kind, is P + W,
in levers of the second...... W − P,
in levers of the third ...... P − W.

If the weight of the lever itself be regarded, it must be added to or subtracted from the power, according as it acts concurrently or non-concurrently with the power. Then, if w = the weight of the lever, the pressure upon the fulcrum,

in levers of the first kind, is $P + W \pm w$,
in levers of the second, is... $W - (P \pm w)$,
in levers of the third, is ...$(P \pm w) - W$.

74. An ordinary balance is, mechanically considered, simply a lever of the first kind, provided with pans suspended from its extremities. When the arms of the balance are equal, a body placed in one of the pans will be balanced by an equal weight in the other. When, however, the arms are unequal, the weight in one pan is not equal to that of the body in the other.

75. *To determine the true weight of a body by a false balance.*

Let w be the true weight of the body, and let x and y be the unknown arms of the balance. When w is placed in one pan, let it be balanced by a weight a; and when placed in the other, by a weight b. Then, since w and a, acting at the arms x and y, were in equilibrium,

$$wx = ay.$$

And since w and b, acting at the arms y and x, were in equilibrium,

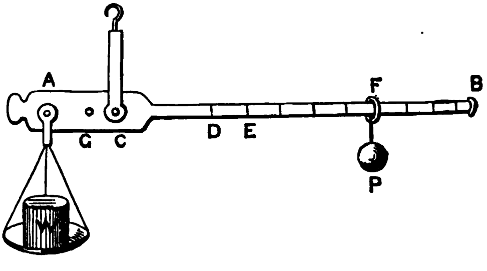
$$wy = bx,$$

$$\therefore \quad w^2xy = abxy,$$

$$\text{hence,} \quad w^2 = ab, \text{ or } w = \sqrt{ab}$$

That is, the true weight is a mean proportional between the two false weights.

76. THE STEELYARD. The common steelyard is a lever of the first kind, with very unequal arms, provided with a moveable weight P, which may be suspended from any point in the longer arm. The longer arm is gra-



duated, and the weight of any body suspended from the shorter arm is indicated by the point at which P must be placed in order to balance it.

77. *To graduate the common steelyard.*

First. When the centre of gravity of the instrument (including both the beam and the scale-pan) is in the shorter arm.

Let G be the centre of gravity, let D be the point at which P must be placed, in order to balance the instrument only, and E the point at which P balances the instrument, when a weight of 1 lb. is placed in the scale-pan; then, if successive distances, each equal to DE, be marked along the arm, they will denote the positions of P, corresponding to successive additions of one pound to the weight.

For, let w be the weight of the instrument; then, since the instrument is balanced by P at D,

$$P.CD = w.GC \ldots\ldots\ldots\ldots\ldots(\text{i.})$$

Again, since a weight of 1 lb. in the scale-pan is balanced by P at E, $\quad P.(CD + DE) = w.GC + AC$;

whence, by subtraction,

$$P.DE = AC \ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.})$$

Let a weight of n lbs. in the scale-pan be balanced by P at F, then $\quad P(CD + DF) = w.GC + n.AC$;

whence, by equations (i.) and (ii.),

$$P(CD + DF) = P.CD + nP.DE;$$

and, therefore, $\quad DF = n.DE$;

that is, if n be 2, $DF = 2DE$; if n be 3, $DF = 3DE$, and so on.

Secondly; when the centre of gravity of the instrument is in the larger arm.

Let G be the centre of gravity, falling now to the right of C.

In this case the steelyard will serve to weigh those bodies only whose weight is greater than $\frac{w.GC}{AC}$. Let p be some whole number of pounds greater than $\frac{w.GC}{AC}$, and let p lbs. in the scale-pan be balanced by P at D, and let $(p + 1)$ lbs. be balanced by P at E;

then, as before, successive distances, each equal to DE, will deno successive increments of 1 lb. in the weight. For, by hypothesi

$$P\,.\,CD + w\,.\,GC = p\,.\,AC \;\ldots\ldots\ldots\text{(i.)}$$

and $$P\,.\,CE + w\,.\,GC = (p+1)\,AC\,;$$

$$\therefore \quad P\,.\,DE = AC \;\ldots\ldots\ldots\ldots\text{(ii.)}$$

Let a weight of $(p+n)$ lbs. be balanced by P at F, then

$$P\,(CD + DF) + w\,.\,GC = (p+n)\,AC\,;$$

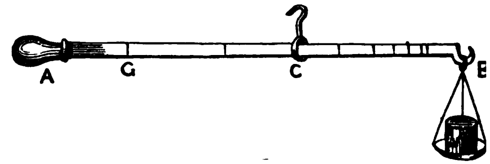
whence, subtracting (i.)

$$P\,.\,DF = n\,.\,AC\,;$$

and, therefore, by (ii.)

$$DF = n\,.\,DE.$$

78. THE DANISH BALANCE. This instrument consists of a straigh bar AB, having a heavy knob A at one end, and at the other end B a hook bearing a scale-pan. The fulcrum C is moveable, and the bar is so graduated, that the weight of an body placed in the scale-pan is determined by the position of C



79. *In the Danish Balance the distances of the fulcrum fro the end of the bar, corresponding to successive equal increments o the weight, form a harmonic series.*

Let w be the weight of the instrument, and G its centre o gravity. Let C be the position of the fulcrum when the weigh of the instrument balances a weight of n units placed in the scale pan. Then

$$nBC = w\,.\,GC,$$
$$= w\,(BG - BC)\,;$$

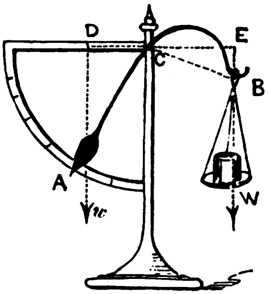
therefore, $$BC = \frac{w\,.\,BG}{w+n}.$$

Consequently, the successive values of BC, corresponding to th values 0, 1, 2, 3, &c. given to n, are

$$\frac{w\,.\,BG}{w},\ \frac{w\,.\,BG}{w+1},\ \frac{w\,.\,BG}{w+2},\ \frac{w\,.\,BG}{w+3},\ \text{\&c.}$$

The reciprocals of these quantities form an arithmetical series, and therefore the quantities themselves are in harmonical progression.

80. THE BENT LEVER BALANCE. This instrument, which is very convenient for determining expeditiously the weight of bodies within a moderate range, consists of a bent lever ACB, one end of which A bears an index, which moves over a graduated quadrant, and the other end B sustains the scale-pan. The thickness of the lever is greatly increased towards A, so that its centre of gravity shall be not far from A. The quadrant may be graduated experimentally, by placing different weights successively in the pan, and marking off the points at which A rests.



The graduation of the quadrant may be also determined by calculation.

Let θ be the angle between the arm AC and the upright stem, corresponding to any weight W. Let w be the weight of the bent lever, and w' the weight of the scale-pan. Then, taking the moments about C,

$$w \,.\, CD = (W + w')\, CE.$$

But if a be the length of the arm AC, and b that of the arm BC; and, for greater simplicity, let the angle ACB be a right angle. Then $CD = a \sin \theta$, and $CE = b \cos \theta$. Therefore,

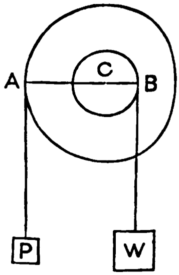
$$w \,.\, a \sin \theta = (W + w')\, b \cos \theta,$$

$$\therefore \qquad \tan \theta = \frac{(W + w')\, b}{w\, a}.$$

As θ approaches a right angle, its tangent increases rapidly; and, consequently, a large increase of the weight will then occasion but a small increase in the angle, and hence the balance cannot be depended on for weights above a certain value.

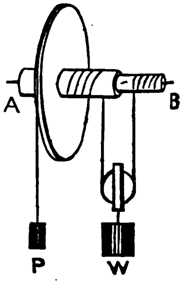
81. The Wheel and Axle. The wheel and axle consists of a cylinder or axle firmly fixed to a wheel, and having a common axis with it. The weight is attached to a cord passing round the axle, and the power to a cord passing round the wheel.

Let the figure represent a section of such a machine, where C is a point in the common axis, CA the radius of the wheel, and CB the radius of the axle. Let P be the power, and W the weight. These may be regarded as two parallel forces, and if in equilibrium, their resultant must pass through C, and therefore we must have P × AC = W × BC, or P × radius of wheel = W × radius of axle. Hence the mechanical advantage in the wheel and axle is found by dividing the radius of the wheel by the radius of the axle.



The wheel and axle admits of various modifications in form. Instead of the wheel, the power may be applied by means of a winch, as in the windlass; or by means of bars inserted into holes pierced in the axle, as in the capstan. In these cases we have, instead of the wheel, one or more of its radii, and the mechanical advantage is still as before, the radius at which the power acts, divided by the radius at which the weight acts.

82. The advantage in the wheel and axle may be increased either by increasing the radius of the wheel, or by diminishing the radius of the axle. If, however, the wheel be very greatly increased, the machine becomes too unwieldy to be serviceable, and if the axle be much diminished, it becomes too weak to sustain the weight. These difficulties in the way of an indefinite increase of the mechanical advantage are overcome by the simple device of a compound axle, one part of which is of smaller radius than the other. One end of



e cord, sustaining the weight, is wound round the thicker part f the axle, and the other end, in a contrary direction, round the hinner part. As the power descends, some part of the cord unwinds from the thinner axle, while another part is wound up around the thicker axle; but as more of the cord is wound up than is let out, the weight is raised by the action of the machine. Let P be the power, W the weight, r the radius of the wheel, a the radius of the thicker axle, and b the radius of the thinner axle. Since the whole weight is supported by the two parts of the cord, the tension in the cord $= \frac{1}{2}\text{W}$.

The power and the tension in the cord passing to the thinner axle both act on the one side of the axle, and the tension in the cord passing to the thicker axle acts on the other. By Art. 43, there will be equilibrium, if the sum of the moments of the former about a point in the axis equals the moment of the latter;

$$\therefore \quad \text{P}r + \tfrac{1}{2}\text{W}b = \tfrac{1}{2}\text{W}a,$$

$$\text{or,} \quad \text{P}r = \tfrac{1}{2}\text{W}\,(a-b);$$

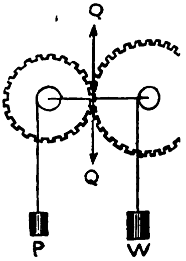
$$\therefore \quad \text{P} : \text{W} :: a-b : 2r.$$

Whence it appears, that the mechanical advantage is equal to twice the radius of the wheel divided by the difference of the radii of the axles. Part of this advantage is owing to the introduction of the pulley, which, it will be presently seen, doubles the advantage of the machine. Consequently, the advantage of the wheel and axle alone is equal to the radius of the wheel divided by the difference of the radii of the axles, or the machine is equivalent to a simple wheel and axle, having an axle equal to the difference between the thicker and thinner parts of the compound axle.

83. Toothed Wheels. A toothed wheel is a circular plate of wood or metal, having its circumference indented or cut into equal teeth all the way round. If two such wheels, having their teeth of the same magnitude, and at the same distance apart, be so placed that a tooth of one may lie between any two of the other; then, if one of them be turned round by any means, the other will be turned round also.

If the teeth be large, the amount and direction of the pres which the one wheel exerts upon the other will, unless the t be made of a peculiar shape, vary considerably as the wh revolve; but if the teeth be small in comparison with the siz the wheel, this variation may be disregarded, and the mu pressure may, without any great error, be treated as constant magnitude and direction.

84. *To find the ratio of the power and weight in toothed wh when the teeth are small.*



Let the power and the weight both act at axles of equal radius, and let c be this radius. Let a be the radius of P's wheel, and b the radius of W's wheel.

Since the teeth are small, the pressure of the one wheel upon the other may be regarded as constant, and as acting in the direction of the common tangent to the two wheels. Let Q denote this pressure. Then, since P and Q are in equilibrium, $Pc = Qa$.

Similarly, $Wc = Qb$;

$\therefore$ multiplying crosswise, $PQbc = WQac$;

$\therefore$ $Pb = Wa$,

or, $P : W :: a : b$.

But since the teeth in each wheel are of the same magnitude at the same distance apart, the number of teeth in each wl will be proportional to the circumference, and consequently to radius. Therefore, P : W : : number of teeth in P's whe number of teeth in W's wheel.

85. THE PULLEY. A pulley consists of a small wheel, wh moves freely about an axis, and allows a cord to pass over any of its circumference. Unless it be otherwise stated, the whee supposed to revolve without friction, and the cord to be perfe flexible.

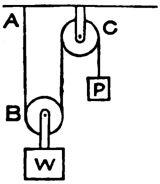
With a single fixed pulley, there is neither gain nor loss of power; for, as the tension in every part of the cord is the same, if a weight W be suspended at one extremity, an equal weight must be applied at the other to maintain equilibrium. Hence, in this case,

$$P = W.$$

The effect of a fixed pulley is simply to change the direction of a force.

86. *To find the ratio of the power and weight in a single moveable pulley, when the cords are parallel.*

Let the weight W be attached to the moveable pulley B, and let B be sustained by a cord ABC, one extremity of which is fastened at A, and the other, after passing over the fixed pulley C, sustains the power P.



Let there be equilibrium; then the weight W is sustained by the tension in BA and the tension in BC. But since the cords are parallel, these tensions may be regarded as two parallel forces, and therefore W must equal their sum. But the tension of the cord is the same throughout, and is equal to P;

$$\therefore \qquad W = 2P, \text{ or } P = \tfrac{1}{2}W.$$

87. The following combinations of pulleys are termed respectively, the first, second, and third system of pulleys.

In the first system of pulleys, each pulley hangs by a separate cord, one end of which is fastened to a fixed beam, and the other to the pulley above it. (See fig. Art. 88.)

In the second system of pulleys, the same cord passes round all the pulleys, which are arranged in two blocks, one of which is fixed, and the other bears the weight. (See fig. Art. 90.)

In the third system of pulleys, each cord is attached to the weight. (See fig. Art. 91.)

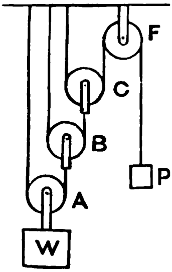
88. *To find the ratio of the power and weight in the first sys of pulleys, when the weight of the pulleys is disregarded.*

Let A, B, C be three moveable pulleys, and let P and W b equilibrium.

By article 86,

$$\begin{aligned}\text{tension in } AB &= \tfrac{1}{2}W,\\ \text{tension in } BC &= \tfrac{1}{2}\text{ tension in } AB = \tfrac{1}{4}W,\\ \text{tension in } CF &= \tfrac{1}{2}\text{ tension in } BC = \tfrac{1}{8}W.\end{aligned}$$

But, by Art. 85, tension in $CF = P$, therefore, when there three moveable pulleys, $P = \frac{1}{8}W$ or $W = 8P$. Similarly, if there be four pulleys, $W = 16P$; if five, $W = 32P$, and so on. Hence, in the first system of pulleys, when the weight of the pulleys is disregarded, the weight is found by doubling the power as many times as there are pulleys, and the power is found by halving the weight as many times as there are pulleys.



Or, if n be the number of moveable pulleys,

$$P = \frac{W}{2^n}, \text{ or } W = 2^nP.$$

89. *To find the ratio of the power and weight in the first sys of pulleys, when the weight of the pulleys is regarded.*

The total weight acting at each pulley is the tension of the st attached to the block, together with the weight of the block.

The entire weight at A is the weight W, together with the wei of the pulley A; one-half of this sum will give the tension in

The entire weight at B is the tension in AB, together with weight of the pulley B; one-half of this sum will give the tens in BC.

The entire weight at C is the tension in BC, together with weight of the pulley C; one-half of this sum will give the tens in CF, which is equal to the power P.

Hence, when W is given to find P: to W add the weight of the lowest pulley, and divide by 2; add the weight of the next pulley, and again divide by 2. Repeat this process as many times as there are moveable pulleys, the result will give P.

If P be given, W is found by the inverse processes. Double P, and subtract the weight of the highest pulley. Double again, and subtract the weight of the next pulley, and so on as many times as there are moveable pulleys.

If n be the number of moveable pulleys, whose weights are severally w_1, w_2, &c...w_n,

$$P = \frac{W + w_1}{2_n} + \frac{w_2}{2^{n-1}} + \&c. + \frac{w_n}{2},$$

$$\frac{1}{2^n}(W + w_1 + 2w_2 + \&c. + 2^{n-1}w_n).$$

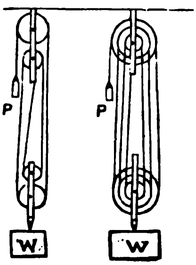
Hence, if all the pulleys be of the same weight, w,

$$P = \frac{W + (2^n - 1)w}{2^n}, \text{ or } W = 2^nP - (2^n - 1)w,$$

$$\text{and } \therefore \quad W - w = 2^n(P - w).$$

90. *To find the ratio of the power and weight in the second system of pulleys.*

Since there is but one cord, and P is attached to one extremity of it, the tension in every part is equal to P. Hence, if n be the number of the portions of this cord in contact with the lower block, the weight supported will be nP; therefore, in this system,

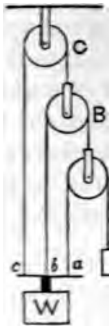


$$W = nP, \text{ or } P = \frac{1}{n}W.$$

If the weight of the lower block be w, the total weight supported is $W + w$; therefore, if the weight of the block be regarded,

$$W + w = nP, \text{ or } P = \frac{1}{n}(W + w).$$

91. *To find the ratio of the power and weight in the third syst of pulleys, when the weight of the pulleys is disregarded.*



Let there be three pulleys A, B, C; then

tension in $Aa = P$,
tension in Bb = pressure on $A = 2P$,
tension in Cc = pressure on $B = 4P$.

But as the weight is supported by the tensions in the three cords Aa, Bb, and Cc, and since the cords are parallel, W must equal the sum of the tensions; therefore,

$$W = P + 2P + 4P = 7P.$$

Similarly, if there be n pulleys,

$$W = P + 2P + 2^2P + \&c. + 2^{n-1}P,$$
$$= (2^n - 1)P.$$

92. *To find the ratio of the power and weight in the third syst of pulleys, when the weight of the pulleys is regarded.*

The tension in Aa (fig. Art. 91) equals the power. The tensi in Bb equals twice the power increased by the weight of t lowest pulley. The tension in Cc equals twice the tension in I increased by the weight of the second pulley. The weight W w be equal to the sum of the three tensions.

Hence generally, add together as many terms of the followi series as there are pulleys; viz., the power, twice the power + t weight of the lowest pulley, twice the preceding + the weight the next pulley, and so on; this sum will be equal to the weig supported.

If there be n pulleys, whose weights, beginning with the lowe are severally w_1, w_2, &c.... w_n, then

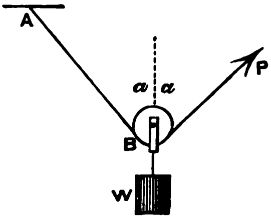
$$W = (2^n - 1)P + (2^{n-1} - 1)w_1 + (2^{n-2} - 1)w_2 + \&c. \dots + w_{n-1}.$$

Or, if all the moveable pulleys be of the same weight w,

$$W = (2^n - 1)\ (P + w) - nw,$$

and $\therefore \quad W + nw = (2^n - 1)\ (P + w).$

93. *To find the ratio of the power and weight in the single moveable pulley, when the cords are not parallel.*



Let 2α be the angle between the cords. The pulley is at rest under the action of three forces; namely, the weight W and the tension in the two parts of the cord, which tension is equal to P. Then, by Art. 22,

$$P : W :: \sin\alpha : \sin 2\alpha :: 1 : 2\cos\alpha;$$

$$\therefore \quad W = 2P\cos\alpha.$$

94. Hence, if two moveable pulleys be combined, as in the first system, with the cords inclined at angles 2α and 2β respectively,

$W = 2$[illegible] $P \cos\alpha \cos\beta.$

For, if T be the tension in the cord bearing the lower pulley, then, by the preceding Article,

$$W = 2T\cos\alpha,$$

and $\quad T = 2P\cos\beta;$

therefore, $\quad W = 2^2 . P\cos\alpha\cos\beta.$

And so generally, if there be n moveable pulleys, and the angles between the cords are $2\alpha_1$, $2\alpha_2$, $2\alpha_3 \ldots\ldots 2\alpha_n$, respectively; then,

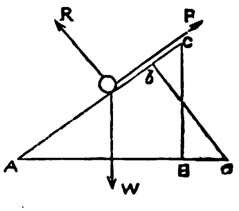
$$W = 2^n\, P\cos\alpha_1, \cos\alpha_2 \ldots\ldots \cos\alpha_n.$$

Hence, if α_1, α_2, &c., be equal to one another,

$$W = P.\,(2\cos\alpha)^n.$$

95. THE INCLINED PLANE. *To find the ratio of the power and weight in the inclined plane, when the power acts parallel to the plane.*

Let W be a weight resting upon the inclined plane AC, and supported by a power P acting parallel to the plane. The forces in equilibrium are gravity, the power P, and the re-action of the plane acting perpendicularly to the plane. Make Ab = AB, and Ac = AC, then the triangle Abc is in all respects equal to the triangle ABC. In the triangle Abc,



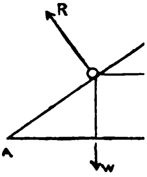
bc is perpendicular to the direction of the power P,
Ac ,, ,, the direction of gravity,
Ab ,, ,, the re-action of the plane;
∴ by Art. 21, P : W : : *bc* : *Ac*,
or, P : W : : BC : AC.

That is, P is to W as the height of the plane is to its lengt mechanical advantage, or the ratio of W to P is, consequen ratio of the length of the plane to the height. In a simil ner the pressure on the plane may be determined. This i and opposite to the re-action of the plane. Let R den re-action,

R : W : : *Ab* : *Ac*,
: : AB : AC.

That is, the pressure on the plane is to W as the base of th is to its length.

96. *To find the ratio of the power and the weight in the plane, when the power acts horizontally.*



In this case W is kept at rest by three forces; viz., the force of gravity acting vertically, the power acting horizontally, and the re-action of the plane acting perpendicularly to the plane. The sides of the triangles ABC are severally perpendicular to the directions of these forces; viz.—

BC is perpendicular to the direction of P,
AB .... W,
AC .... R.

Therefore P : W : : BC : AB; or the power is to the w the height of the plane is to the base.

Hence the mechanical advantage is the ratio of the base height.

Also, R : W : : AC : AB; or the pressure on the plane i weight as the length of the plane is to the base.

97. *Two weights, connected by a cord which passes over at the summit, rest upon a double inclined plane; to find th when in equilibrium.*

Let l and l' be the length of the two planes supporting respectively the weights W and P. Let h be the common height of the planes. If T denote the tension in the cord, then, since W is supported on an inclined plane by a force T acting along the plane, by Article 95,

$$T : W :: h : l;$$

Similarly, $$T : P :: h : l',$$

$$\therefore \quad P : W :: l' : l.$$

That is, the two weights are in the same ratio as the lengths of the planes on which they respectively rest.

98. *To find the vertical and horizontal pressures, when a force acts perpendicularly to a given inclined plane.*

Let R in the figure of Art. 96 represent the given force, W and P vertical and horizontal forces in equilibrium with R, and consequently equal and opposite to the vertical and horizontal pressures exerted by R.

But $W : R :: AB : AC$; therefore,

$$\text{the vertical pressure} = R \times \frac{\text{base of the plane}}{\text{length of the plane}}.$$

And $P : R :: BC : AC$; therefore,

$$\text{the horizontal pressure} = R \times \frac{\text{height of the plane}}{\text{length of the plane}}.$$

99. *To find the pressure upon an inclined plane in terms of the power and the mechanical advantage.*

Let P be the power and a the mechanical advantage; then the weight $= aP$. Then, if the power acts along the plane, the weight equals the resultant of the power and the pressure; and therefore, since the latter act at right angles,

$$R^2 + P^2 = W^2 = a^2 P^2;$$

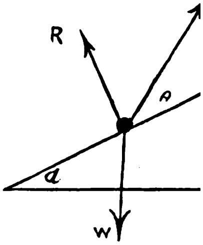
$$\therefore \quad R = P\sqrt{(a^2 - 1)}.$$

If the power act horizontally, the pressure equals the resultant of the power and weight; and therefore, since these act at right angles,

$$R^2 = P^2 + W^2 = P^2 + a^2 P^2;$$

$$\therefore \quad R = P\sqrt{(a^2 + 1)}.$$

100. *To find the ratio of the power and the weight in the plane, when the power acts in any direction whatever, and find the pressure on the plane.*



Let the power be inclined to the plane at an angle θ. Then, by the triangle of forces,

$$P : W :: \sin \widehat{RW} : \sin \widehat{PR}.$$

But $\widehat{RW} = 180^\circ - \alpha$, and $\widehat{PR} = 90^\circ - \theta$; therefore,

$$P : W :: \sin \alpha : \cos \theta,$$

or, $$W = \frac{P \cdot \cos \theta}{\sin \alpha}.$$

To find the pressure on the plane, we have

$$R : W :: \sin \widehat{PW} : \sin \widehat{PR};$$

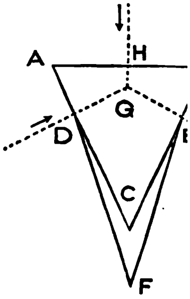
but $\widehat{PW} = 90^\circ + \alpha + \theta$; therefore,

$$R : W :: \cos(\alpha + \theta) : \cos \theta,$$

or, $$R = \frac{W \cdot \cos(\alpha + \theta)}{\cos \theta}.$$

101. THE WEDGE. *To find the ratio of the power and sistance in an isosceles wedge.*

Let ABC be the section of an isosceles wedge introduc the cleft DFE, and let the points DE be similarly situate on the two sides of the wedge. The resistance on each side of the wedge will be the same, and if R = the total resistance, the resistances at D and E will each $= \frac{1}{2}R$; and they act perpendicularly to the sides of the wedge. Let a power P act at the point H, the centre of the back of the wedge. The directions of these three forces when produced will meet in a point G; they may, therefore, be considered as three forces



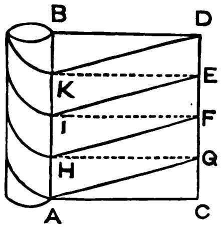
upon a point, and in equilibrium. The sides of the triangle ABC are severally perpendicular to the directions of the three forces, and therefore,

$$P : \tfrac{1}{2}R :: AB : AC;$$

$$\therefore \quad P : R :: \tfrac{1}{2}AB : AC.$$

That is, the power is to the total resistance, as half the back of the wedge is to the side of the wedge.

102. The Screw. Let AB be a cylinder, and AD a rectangle, whose base AC is equal to the circumference of the cylinder. Let the side CD be divided into any number of equal parts. Join AG, and through the points F, E, D, draw FH, EI, DK parallel respectively to AG. Then, if the rectangle AD be wrapped round the cylinder, the parallel lines AG, HF, &c. will trace out the continued spiral line called the screw.



Hence, any resistance to be overcome by a screw may be regarded as a weight resting upon an inclined plane, whose base is equal to the circumference of the cylinder, and whose height is equal to the distance between the threads of the screw.

When the screw is used for mechanical purposes, the power is always applied in a direction perpendicular to the axis of the cylinder, and consequently parallel to the base of the inclined plane forming the screw line; and, therefore,

P : W : : height of inclined plane : the base;

that is, P : W : : distance between the threads of the screw : circumference of the cylinder.

EXAMPLES.

1. A bar of uniform thickness and density, whose weight is 6 lbs., and length 4 ft. 2 in., is used as a lever of the first kind; the fulcrum is 10 inches from one end of the bar; what power will sustain a weight of 25 lbs.? Ans. 4 lbs.

2. A bar of uniform thickness and density, weighing 2 lbs. per foot, is used as a lever of the first kind; the fulcrum is 9 inches from one end of the bar; what must be the length of the bar when its weight balances a weight of 36 lbs. suspended at the end of the shorter arm? Ans. 6 ft.

3. In a weightless lever of the first kind, a power P acting at an arm a, sustains the same weight as the power Q acting at an arm b; required the length of the lever, and the weight sustained.

The length of the lever is equal to

$$\frac{ab(P-Q)}{Pa-Qb},$$

and the weight sustained is equal to

$$\frac{Pa-Qb}{b-a}.$$

4. Two men, able to exert forces of 260 and 300 lbs. respectively, work the handle of a winch and axle. The radius of the axle is 5 inches, what must be the length of the arm of the winch that the men may be just able to raise a weight of 2 tons?

Ans. 3 ft. 4 in.

5. A weight of 10 lbs. is suspended from a single moveable pulley, the cords are inclined at an angle of 30°; required the tension in the cord. Ans. $5(\sqrt{6}-\sqrt{2})$ lbs.

6. Five pulleys, each weighing 3 lbs., are arranged according to the first system; what weight will be supported by a power of 20 lbs.? Ans. 547 lbs.

7. In the first system of pulleys, if p be the power necessary to balance the weight of the blocks, and if a be the mechanical advantage of the system when the weight of the blocks is disregarded; then will $W = a(P - p)$.

8. In the first system of pulleys, composed of blocks of equal weight, show that the power required to balance the weight of the blocks approaches more and more nearly the weight of one of them, as the number of blocks is increased.

9. A weight $2P$ is supported on an inclined plane by two equal forces P and P, one acting horizontally and the other along the plane; required the inclination of the plane.

Ans. The plane rises 4 in 5.

10. A weight $2P$ is supported on an inclined plane by two equal forces P and P, one acting horizontally and the other making with the plane an angle equal to the inclination of the plane; required the inclination of the plane and the pressure upon the plane.

Ans. The inclination of the plane is 45°, and the pressure upon the plane is equal to $P\sqrt{2}$.

11. If the arms of a lever of the first kind, of uniform thickness and density, be a and b, and if the weights P and W be in equilibrium, what is the weight of the lever?

Ans. $\dfrac{2(Wb - Pa)}{a - b}$.

12. If the same bar be used as a lever of the second kind, having as before the shorter arm equal to b, and if the weights P and W be in equilibrium when the weight of the bar acts concurrently with W, what is the weight of the lever?

Ans. $2\left(P - \dfrac{Wb}{a + b}\right)$.

13. If P and W be in equilibrium in the simple wheel a axle, and if P be caused to descend, show that P × distan through which P moves = W × distance through which W moves

14. Show the same in the case of the compound wheel and ax

15. If two weights P and W, connected by a cord which pass over a pulley at the summit, are in equilibrium upon a doub inclined plane, show that, when the weights are put in motio P × vertical height through which P moves = W × vertical heig through which W moves.

16. In a single moveable pulley, if the angle between the cor be 60°, show that $W = P\sqrt{3}$.

17. If three pulleys, whose weights are w_1, w_2, and w_3, arranged according to the first system, and P and W be in eq librium, show that if P descend through a space a, and x_1, x_2, be the spaces through which the pulleys severally rise in con quence; then,

$$Pa = Wx_1 + w_1x_1 + w_2x_2 + w_3x_3.$$

18. When a weight is supported upon a plane whose inclinati is α, what angle must the direction of the power make with t plane that the resistance may be equal to the power?

Ans. $90° - 2\alpha$.

19. If the inclination of a plane be 30°, and the direction of t power make with the plane an angle of 30°, show that $W = P\surd$

20. If P and Q, connected by a string passing over a fix pulley, be in equilibrium on a double inclined plane; if α and be the inclination of the planes, and θ and ϕ the angles which t two portions of the string make with the planes; then,

$$\frac{P}{Q} = \frac{\cos\theta}{\cos\phi} \cdot \frac{\sin\beta}{\sin\alpha}.$$

21. A weight $(m+n)$ P is supported upon an inclined plane by two forces, one mP acting horizontally, and the other nP making an angle with the plane equal to the inclination of the plane; required the inclination of the plane. Ans. 45°.

22. A weight of 13 lbs. is supported upon an inclined plane by a power of 10 lbs., making an angle with the plane equal to half the inclination of the plane; required the inclination of the plane.

Ans. $\tan^{-1}\left(\frac{120}{129}\right)$.

23. A cord fastened at A passes under a moveable pulley bearing a weight P, it then passes over a fixed pulley at B, and passing under a second moveable pulley bearing a weight Q, is finally fastened to a fixed peg at C; required the tension in the cord when the angle at one of the moveable pulleys is double the angle at the other.

Let P be greater than Q, then the tension required is equal to

$$\frac{\sqrt{(Q^2+8P^2)}-Q}{4}.$$

24. Find the pressure on the axis of a winch and axle.

Let P be the power, a the mechanical advantage, and θ the angle between the arm of the weight and the arm of the power, then the pressure on the axis is equal to

$$P\sqrt{(a^2-2a\cos\theta+1)}.$$

CHAPTER V.

ON COMBINATIONS OF THE SIMPLE MACHINES.

103. *To find the mechanical advantage of any combination of machines.*

In a combination of machines, the weight of the first machine is the power of the second, the weight of the second the power of the third, and so on. Let a, b, c, be the separate advantages of three machines in combination. Let P and Q be the power and weight in the first, Q and R those in the second, R and W those in the third; then,

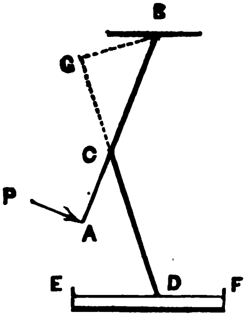
$$
\begin{array}{ll}
 & Q = aP, \\
\text{and} & R = bQ, \\
\text{and} & W = cR. \\
\text{Multiplying together} & QRW = abc\ PQR; \\
\therefore & W = abc\ P.
\end{array}
$$

Therefore abc represents the mechanical advantage of the combination; that is, the advantage of the combination is equal to the product of the separate advantages of the component machines.

In like manner it may be shown that the mechanical advantage of the combination of any number of machines is equal to the product of the separate advantages of the component machines.

104. **The Knee Lever.** This is a combination of levers, used in some forms of the printing press, and in other cases where a very great pressure is required to act through a very small space. It consists of a bar ABC, moveable about a fixed pivot at B, and fastened to a second bar at C. The bar CD is connected by a pivot at D with a moveable plate EF. The lengths CB and CD

are so taken that the two bars are nearly in the same straight line, when the plate EF is in its lowest possible position. The power is applied at A, at right angles to AB, and the plate is in consequence pressed down with a very great force.



Let W be the thrust transmitted along the bar CD; then, since CD is nearly perpendicular to the plate when the instrument is in operation, W will very nearly represent the effective pressure on EF.

To determine the ratio of W to P, draw BG at right angles to DC produced. The bar AB is in equilibrium under the action of the force P, and a force equal and opposite to the thrust in CD, that is, a force W acting in the direction CG; wherefore, taking their moments about the fixed point B, we have,

$$W \times BG = P \times AB;$$

$$\therefore \qquad W = P \times \frac{AB}{BG}.$$

But when the bars are nearly in the same straight line, BG is very small, and consequently $\frac{AB}{BG}$ is very large.

105. If the larger arm of AB be a, and the shorter b, and if, when the instrument is in operation, the angle ACD be θ, then

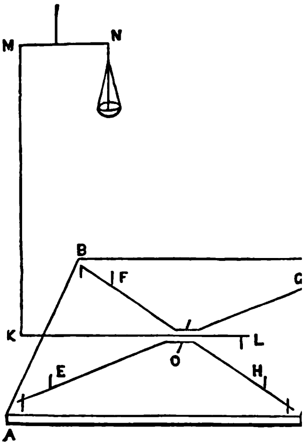
$$W = \frac{Pa}{b \sin \theta},$$

and therefore the mechanical advantage is equal to

$$\frac{a}{b \sin \theta}.$$

106. THE WEIGHING MACHINE. This is a combination of levers, so arranged as to furnish a convenient means for determining the weight of carriages and other heavy bodies. In a rectangular framework ABCD are placed four equal levers of the

second order, having their fulcra at A, B, C, and D, an other extremities connected together at O, and also fastene by a pivot to a lever KL. This lever is also of the second order, having its fulcrum at L, and is connected by a rod KM with a small balance MN. At E, F, G, and H, are four pins, severally equidistant from A, B, C, and D. Upon these pins a platform (not represented in the figure) rests, and upon the platform the body to be weighed is placed. When used for weighing carriages, the framework is sunk below the road, so that the platform may be on a lev the road. In place of the balance MN, a small steel frequently used.



In determining the mechanical advantage of this mach must be observed, that the four levers, AO, BO, CO, DO, in combination, but are simply a contrivance for the con support of the weight. The mechanical advantage is simp of one of them. For let W be the entire weight on the pl and let V, X, Y, and Z, be the parts of it, acting severally points E, F, G, and H, and let m be the advantage of each four levers; then,

V at E will be balanced by $\frac{V}{m}$ at O,

X at F will be balanced by $\frac{X}{m}$ at O,

Y „ G „ „ $\frac{Y}{m}$ „ O,

Z „ H „ „ $\frac{Z}{m}$ „ O.

Hence, the entire pressure at O is equal to

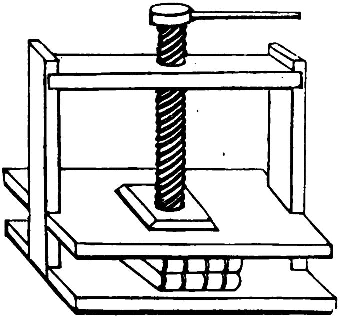
$$\frac{V+X+Y+Z}{m}; \qquad \text{that is, is equal to } \frac{W}{m}.$$

Hence, the instrument is equivalent to a combination of only three levers. Let A and a be the lengths of the longer and shorter arms of the lever AO, B and b those of the lever KL, and C and c those of the lever MN; then,

$$\text{mechanical advantage} = \frac{A}{a} \times \frac{B}{b} \times \frac{C}{c},$$

$$= \frac{ABC}{abc}.$$

107. THE SCREW PRESS. This instrument is a combination of the screw and the lever, the power being applied at the extremity of a lever, connected with the screw at right angles with its axis. The fulcrum of the lever is at the axis of the screw, and the pressure exerted by it is at the circumference of the screw. The lever is consequently a lever of the second kind, having its shorter arm equal to the radius of the screw. Let r be the radius of the screw, d the distance between the threads, and l the distance of the point at which the power acts from the axis of the screw; then,



$$\text{advantage of lever} = \frac{l}{r},$$

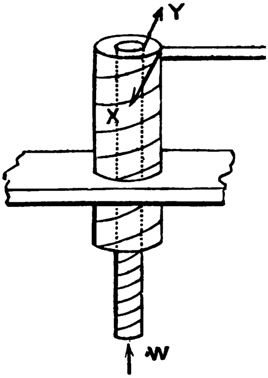
$$\text{advantage of screw} = \frac{2\pi r}{d};$$

therefore,

$$\text{advantage of press} = \frac{2\pi l}{d}.$$

Hence, the advantage of the press is equal to the circumference described by the power divided by the distance between the threads.

108. **Hunter's Screw.** In the common screw press, tl chanical advantage is increased either by increasing the of the lever, or by diminishing the distance between the threads. Practically, however, the increase cannot be continued indefinitely by either method. If the length of the lever be too great, the instrument is unwieldy; and if the threads be too fine, they will not be strong enough to sustain the pressure exerted upon them. These difficulties are overcome by the screw invented by Mr. Hunter, and called after his name.



Instead of the single screw of the common screw press, he substituted two screws, one of worked within the other. The larger screw, as in the common works through a nut; and the smaller screw is fastened moveable plate. The distance between the threads of the s screw is less than that between the threads of the larger scre

To determine the mechanical advantage of the combinati P be the power applied at the end of the lever, and let W 1 resistance sustained by it.

Resolve P into two forces X and Y, acting respectively circumferences of the two screws; then, if l be the distance from the common axis of the screws, r and r' the radii screws, by Art. 42, $Pl = Xr - Yr'$.........(i.)

But since a weight W is sustained by a power X acting circumference of the larger screw, if d be the distance betwe threads, $X = \frac{Wd}{2\pi r}$...............(ii.)

In like manner, if d' be the distance between the thre the smaller screw, $Y = \frac{Wd'}{2\pi r'}$ ....(iii.)

Substituting in equation i. the values given by ii. and iii. then

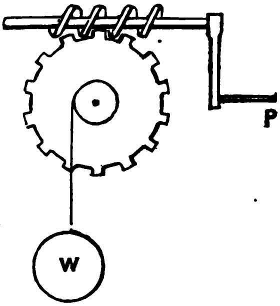
$$Pl = \frac{Wd}{2\pi} - \frac{Wd'}{2\pi};$$

therefore,

$$\frac{W}{P} = \frac{2\pi l}{d-d'};$$

or the mechanical advantage is the circumference described by the power divided by the difference in the intervals between the threads of the larger and smaller screw.

109. THE ENDLESS SCREW. This instrument is a combination of the screw, and the wheel and axle. The screw is worked by a winch; and the threads of the screw, instead of working in a nut, press against the teeth formed on the circumference of the wheel.



Let l be the arm of the winch, d the distance between the threads of the screw, R the radius of the wheel, and r the radius of the axle; then, by Art. 107,

$$\text{advantage of winch and screw} = \frac{2\pi l}{d};$$

and, by Art. 81,

$$\text{advantage of wheel and axle} = \frac{R}{r},$$

therefore,

$$\text{advantage of endless screw} = \frac{2\pi l}{d} \cdot \frac{R}{r}.$$

Let n be the number of teeth in the circumference of the wheel, then, since the teeth of the wheel are to work in the screw,

$$\frac{2\pi R}{n} = d,$$

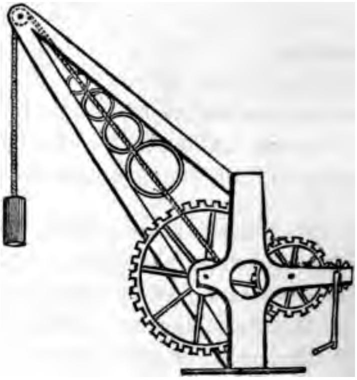
and, therefore,

$$\frac{2\pi R}{d} = n;$$

consequently,

$$\text{advantage of endless screw} = \frac{nl}{r}.$$

110. **The Crane.** This machine, used for raising and lowering heavy weights, is a combination of wheels and axles. A common form is represented in the figure. This consists of a winch and axle, and of two wheels and axles. The axle of the winch, and the axle of the first wheel, are furnished with teeth,* as are also the two wheels. The power is applied at the handle of the winch, and the weight is sustained by a rope or chain passing round the axle of the second wheel.



Let l be the length of the arm of the winch, and a the radius of its axle, then

$$\text{advantage of winch and axle} = \frac{l}{a}.$$

Let R_1 be the radius of the first, and r_1 the radius of its axle, then

$$\text{advantage of first wheel and axle} = \frac{R_1}{r_1}.$$

Let R_2 be the radius of the second wheel, and r_2 the radius of its axle, then

$$\text{advantage of second wheel and axle} = \frac{R_2}{r_2}.$$

Consequently,

$$\text{advantage of crane} = \frac{l\, R_1\, R_2}{a\, r_1\, r_2}.$$

If N_1, N_2, be the number of teeth respectively in the two wheels, and n_1, n_2, those in the axles of the winch and of the first wheel, then

$$\frac{R_1 \,.\, R_2}{a\, r_1} = \frac{N_1\, N_2}{n_1\, n_2},$$

and, therefore,

$$\text{advantage of crane} = \frac{l \,.\, N_1\, N_2}{n_1 \,.\, n_2 \,.\, r_2}.$$

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* A toothed axle is commonly called a *pinion*, and its teeth are termed *leaves*.

111. The Principle of Virtual Velocities. If in any machine the power and weight be in equilibrium, and the machine be put in motion without disturbing the equilibrium, then the spaces described by the two forces are as the forces inversely, the spaces being measured in the direction of the forces respectively; that is to say,

$$\frac{\text{space described by P}}{\text{space described by W}} = \frac{\text{W}}{\text{P}}.$$

This may be easily verified in the case of the simple machines.

Take, for instance, the wheel and axle. Let the wheel make an entire revolution, then the space through which P (fig. Art. 81) will move, will be that of the length of cord given out (or taken in) by the wheel; that is, will be equal to the circumference of the wheel: and the space through which W will move will be that of the length of cord taken in (or given out) by the axle; that is, will be equal to the circumference of the axle. Hence in this case the space described by P is to the space described by W as the circumference of the wheel is to the circumference of the axle. But the circumferences of circles are as their radii; hence

$$\frac{\text{space described by P}}{\text{space described by W}} = \frac{\text{radius of wheel}}{\text{radius of axle}}.$$

But by Art. 81

$$\frac{\text{W}}{\text{P}} = \frac{\text{radius of wheel}}{\text{radius of axle}},$$

$$\therefore \quad \frac{\text{space described by P}}{\text{space described by W}} = \frac{\text{W}}{\text{P}}.$$

If now, instead of causing the wheel to make an entire revolution, it make any part of a revolution, say one-tenth, the spaces described by P and W will be one-tenth of the circumference of the wheel, and one-tenth of the circumference of the axle, and these have the same ratio as before; and so, whatever the distance through which the wheel may revolve, P's space is to W's space as the radius of the wheel is to the radius of the axle; that is, as W is to P.

Again: take the first system of pulleys, and let there be three moveable pulleys as in Art. 88. Let W be raised through a space, say 1 inch. The lowest block will then be raised 1 inc and 2 inches of cord will be given off from A; the second blo will then be raised 2 inches, and 4 inches of cord will be given from B; the third block will then be raised 4 inches, and 8 inch of cord will be given off from C, and this consequently will be t space over which P will descend. Hence the space described by is 8 times that described by W; and this is the ratio of W to P.

In like manner the principle may be verified with any numb of moveable pulleys.

Again: take the inclined plane when the power acts along th plane. Let the weight move along the plane any distance, and le AC (fig. Art. 95) represent this distance. Then the distanc through which P moves in its own direction is AC; but W, ac ing vertically, moves in its own direction only through the distanc BC. Hence P's space is to W's space as AC is to BC; that is, a the length of the plane is to the height, and consequently as W i to P.

The student is recommended to verify the principle by simila methods in the cases of the other simple machines.

112. By means of the principle of virtual velocities the mechani cal advantage of any machine may often be readily found. If i any one instance the spaces described by P and W in their ow direction can be found, the ratio of these spaces, being the sam for all instances, gives us the ratio of W to P; that is, gives u the mechanical advantage.

Take, for example, the screw press. If the lever (see fig. Ar 107) make an entire revolution, it is clear that the screw will mov forward through the distance between the threads. Hence if l b the length of the lever, and d the distance between the threads,

$$\text{mechanical advantage} = \frac{2\pi l}{d}.$$

Again: take the case of the endless screw. If the winch (se fig. Art. 109) be turned round once one of the teeth of the whe is moved forward; and hence if the winch be turned round as many times as there are teeth in the wheel, the wheel, and consequently the axle, will make one entire revolution. Hence if n be the number of teeth, l the length of the winch, and r the radius of the axle, while P moves through n times $2\pi l$, W moves through $2\pi r$, and therefore

$$\text{mechanical advantage} = \frac{2\pi l n}{2\pi r} = \frac{ln}{r}.$$

EXAMPLES.

1. In a screw press, the length of the lever is l; what must be the distance between the threads of the screw in order that the greatest possible mechanical advantage of the combination may be A? Ans. $2\pi l \div \text{A}$.

2. In a weighing machine, the arms of the lever AO (see fig. Art 106) are 8 feet and 1 foot, the arms of the lever KL are 8 feet and 6 inches, and the arms of the balance are 14 inches and 1 inch; what weight on the platform will be sustained by 1 oz. in the scale-pan? Ans. 1 cwt.

3. In a Hunterian screw, the larger screw has 10 threads to an inch, and the smaller screw has 11 threads to an inch, and the length of the lever is 10 inches; required the mechanical advantage. Ans. 6911·5 nearly.

4. In the common screw press, determine the pressure upon the threads when a power P acts at the extremity of the lever.

If a be the advantage of the press, and b that of the lever, then the required pressure is equal to $\text{P}\sqrt{(a^2 + b^2)}$.

5. In a crane, the number of teeth in the two wheels are 30 and 40, the number of leaves in the two pinions are 6 and 10, the arm of the winch is 2 feet, and the radius of the weight-bearing axle is 3 inches; required the mechanical advantage. Ans. 160.

6. Determine the pressures on the three axes of a crane when the power and the weight are both at right angles with the line joining the axes.

Let a be the advantage of the winch and axle, then the pressure on their axis is equal to $$P(a+1);$$
let b be the advantage of the first wheel and axle, then the pressure on the axis is $$Pa(b+1);$$
and let c be the advantage of the second wheel and axle, then the pressure on their axis is $$P\,ab\,(c+1).$$

7. If a weight be raised by an endless screw, the mechanical advantage of which is a, show that the space described by the power is equal to a times that described by the weight.

8. In a weighing machine, the advantage of the lever AO (see fig. Art. 106) is a, the advantage of the lever KL is b, and the advantage of the lever MN is c; determine the pressures upon the different fulcra, when a weight W is placed upon the platform so as to press equally upon the four pins E, F, G, H.

The pressures at A, B, C, and D are each equal to
$$\frac{W(a-1)}{4a}.$$
The pressure at L is equal to
$$\frac{W(b-1)}{ab},$$
and the pressure at the fulcrum of MN is equal to
$$\frac{W(c+1)}{abc}.$$

CHAPTER VI.

ON FRICTION.

113. The mutual reactions, when one solid body is in contact th another, have hitherto been regarded as acting entirely in the rection of the common normal through the point of contact. is is true only upon the supposition that the surfaces in contact e perfectly smooth. Upon such hypothesis, the reactions of the rfaces oppose in no degree the motion of one body along the other, ut the slightest conceivable force acting along a common tangent rough the point of contact will move the body in that direction. ince, however, the surfaces of bodies in contact are in no case rfectly smooth, the foregoing result does not obtain in practice. force varying in amount in different cases, but of an appreciable agnitude, is found to be necessary in order to move one body ong the surface of another. To this force the name of *friction* given. The total reaction of the surfaces is therefore not in the irection of the normal, but is inclined to it at a greater or less gle. If the reaction be resolved into two forces, one normal and e other tangential to the surfaces in contact, the latter component ill be the force of the friction.

114. Friction is a retarding force merely; it can destroy motion, ut cannot generate it. It therefore acts upon any body only when tendency to motion exists; that is, when the other forces acting pon the body are not in equilibrium. Also, being a retarding rce, its direction is the opposite of that in which the body tends move. Hence, if in any machine the power be greater than is quired by the conditions of equilibrium, the force of friction will t concurrently with the weight; but if less, it will act concur-

rently with the power. Hence, a given weight may be sustained by a less power than is required independently of friction, but needs a greater power to move it.

115. From various experiments, made with the view of determining the laws of friction, the following general results have been obtained:—

First: *Friction is increased by time.* When two bodies have been for some time in contact, it is found that a greater force is needed to move the one over the other than to keep them moving when in motion. Thus, for instance, when the surfaces are both of oak, the force required to move one over the other, after a contact of some minutes, was found, in the experiments of M. Arthur Morin, to bear to that required to keep them moving the proportion of 62 to 48; and when a surface of iron was made to move over one of oak, the forces were as 65 to 26. The time during which the force of friction continues to increase varies in different materials. When both surfaces are wood, the maximum of friction is reached after a contact of two or three minutes; when both are metals, it is reached almost instantaneously, but when wood is placed upon metal, after a contact of several days.

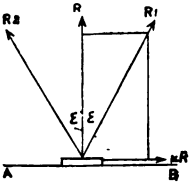
Secondly: *Between the same surfaces, and under the same circumstances, friction is proportional to the pressure.* The force of friction is found to depend upon the relative disposition of the surfaces in contact; as, for instance, whether the fibres be parallel or at right angles, and also upon the state of the surfaces, whether dry or lubricated. If no change be made in the disposition or state of two surfaces in contact, it is found, when the pressure is varied, that the ratio of the friction to the pressure remains constant for the same materials. This constant ratio is termed the *coefficient of friction.* Accordingly, if μ be the coefficient of friction between any two materials, and P be the normal pressure, the friction is equal to μP.

Although considerable variation exists in the values of μ, as determined by different experimentalists, the constancy of its value for the same materials may be regarded as established, both for surfaces which have been for some time in contact, and for surfaces in motion; save only that when the pressure is very large, the force of friction is found to be somewhat less than as given by this law.

Thirdly: *With the same surfaces, and under the same circumstances, friction is independent of the extent of the surface of contact.* This law again is to be regarded as only approximately true. It does not apply when the surfaces in contact are very large or very small. When the surfaces touch each other in a straight line, the friction is less than in the contact of plane surfaces of finite extent; and, on the other hand, the friction is increased when the surfaces are very large.

116. *Def.* The angle whose tangent is equal to μ is, for a reason that will appear presently, termed the *angle of repose.*

117. *When one body in contact with another is in a state bordering on motion, the reactions of the surfaces act in a direction which makes with the normal an angle equal to the angle of repose.*



Let a body resting upon the plane AB be upon the point of moving along the plane towards A; then, if R be the normal resistance of the plane, and μ the coefficient of friction, the force of friction will be a force μR acting towards B. Let R_1 be the resultant of R and μR; then, by the parallelogram of forces, R_1 will be represented by the diagonal of the parallelogram whose sides are proportional to R and μR. Hence, if ϵ be the angle between the directions of R_1 and R,

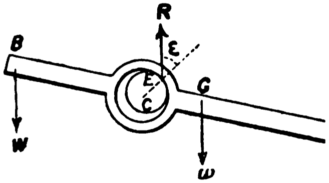
$$\tan \epsilon = \frac{\mu R}{R} = \mu,$$

or ϵ is the angle of repose. In like manner, if the body be on the point of moving towards B, the reaction of the surfaces will act in the direction of R_2 at an angle ϵ on the other side of the normal.

Hence, in questions of equilibrium in which friction i sidered, the reaction of two surfaces in contact may be introduced as a single force acting at the angle of repose on side of the normal, or as two forces R and μR acting respe(along the normal and tangent. The former will be mor venient in cases when not more than two other forces are upon the bodies.

118. *To find the limits of equilibrium in a lever of the firs when the power and weight act in parallel directions.*

Let AB be a lever moving about a fixed cylindrical a: means of a cylindrical hole, whose centre is C, and whose ra(r. Let AC $= a$ and BC $= b$, and let ϵ be the angle of repose the weight of the lever be disregarded, and let P be on the of overcoming W. Let R, the resistance of the pivot, act at E. Then since R is equal and opposite to the resultant of P and W, R is parallel to P and W; and also since the lever is on the point of slipping the direction of R, makes the angle ϵ with the normal at E.] taking moments about E we have,



$$P\,(a - r \sin \epsilon) = W\,(b + r \sin \epsilon).$$

In like manner, if W be on the point of overcoming P, it r shown that

$$P\,(a + r \sin \epsilon) = W\,(b - r \sin \epsilon).$$

If the weight of the bar itself be taken into account, le the centre of gravity, w the weight of the bar, and c the di of G from C. Then if P be on the point of overcomi taking as before the moments about E,

$$P\,(a - r \sin \epsilon) + w\,(c - r \sin \epsilon) = W\,(b + r \sin \epsilon),$$

or, if W be the point of overcoming P,

$$P\,(a + r \sin \epsilon) + w\,(c + r \sin \epsilon) = W\,(b - r \sin \epsilon).$$

Ordinarily the radius of the pivot and the radius of the hollow cylinder are so nearly equal, that the one may be substituted for the other without sensible error.

COR. I. The wheel and axle, when the power and weight act in the same direction, may be regarded as a lever of the first kind, having its centre of gravity at C. Then if a be the radius of the wheel, and b the radius of the axle, the limits of equilibrium (since $c=o$) will be expressed by the equations

$$P\ (a \mp r \sin \epsilon) \mp wr \sin \epsilon = W\ (b \pm r \sin \epsilon),$$

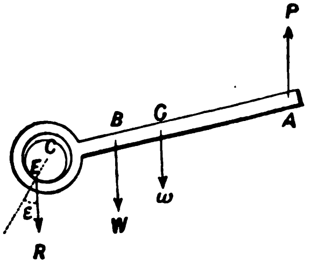
the upper signs being taken when P is on the point of overcoming W, and lower when W is on the point of overcoming P.

COR. II. The single fixed pulley also may be regarded as a lever of the first kind with equal arms, and its centre of gravity at C. Hence, if a be the radius of the pulley, and r the radius of the pivot, the limits of equilibrium for the single fixed pulley with parallel cords will be expressed by the equations,

$$P\ (a \mp r \sin \epsilon) \mp wr \sin \epsilon = W\ (a \pm r \sin \epsilon).$$

119. *To find the limits of equilibrium in a lever of the second kind, the power and weight acting in parallel directions.*

As before, let $AC = a$, $BC = b$, $GC = c$, and let r be the radius of the hollow cylinder.



Let P be on the point of overcoming W, and let the resistance R act at E. Then as before, R is parallel to W and P, and makes an angle ϵ with the normal at E.

Then taking moments about E, we have

$$P\ (a + r \sin \epsilon) = W\ (b + r \sin \epsilon) + w\ (c + r \sin \epsilon).$$

In like manner, if W be on the point of overcoming P, it may be shown that

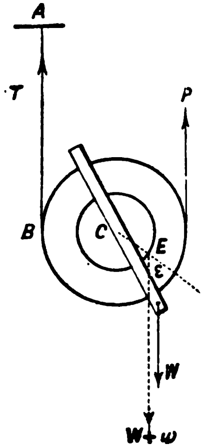
$$P\ (a - r \sin \epsilon) = W\ (b - r \sin \epsilon) + w\ (c - r \sin \epsilon).$$

COR. The wheel and axle, when the power and weight act contrary directions, may be regarded as a lever of the second kin having its centre of gravity at C. Then if a be the radius of th wheel, and b the radius of the axle, the limits of equilibriu (since $c = o$) will be expressed by the equations

$$P(a \pm r \sin \epsilon) = W(b \pm r \sin \epsilon) \pm wr \sin \epsilon,$$

the upper signs being taken when P is on the point of overcoming W, and the lower when W is on the point of overcoming P.

120. *To find the limits of equilibrium in a single moveable pulley.*



Let a be the radius of the pulley, r the radius of the pivot, and w the weight of the block, and let P be on the point of overcoming W.

Let the resultant of W and w meet the pivot at E, the normal at E will then make an angle ϵ with the vertical line. Then if T be the tension in the cord AB, and the moment be taken about C.

$$Pa = Ta + (W + w)\, r \sin \epsilon.$$

But $T = W + w - P$: therefore,

$$2Pa = (W + w)(a + r \sin \epsilon).$$

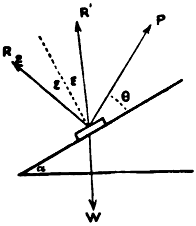
In like manner, if W is on the point of overcoming P, it may be shown that

$$2Pa = (W + w)(a - r \sin \epsilon).$$

121. *To find the limits of equilibrium when a body whose weight is* W *is supported upon an inclined plane by a power acting in a given direction.*

Let α be the inclination of the plane, and let the power mak an angle θ with the plane.

First: Let the body be on the point of moving down the plane, the reaction of the plane will then act in the direction of R_1. Let P_1 be the value of P, in this case, then,



$$P_1 : W : : \sin \widehat{R_1 W} : \sin \widehat{R_1 P},$$

or,
$$P_1 : W : : \sin (\alpha - \epsilon) : \cos (\theta + \epsilon) ;$$

$$\therefore \quad P_1 = W . \frac{\sin (\alpha - \epsilon)}{\cos (\theta + \epsilon)} = W . \frac{\sin \alpha - \mu \cos \alpha}{\cos \theta - \mu \sin \theta}$$

Secondly: Let the body be on the point of moving up the plane, then R_2 will be the reaction of the plane; and if P_2 be the corresponding value of P, then

$$P_2 : W : : \sin \widehat{R_2 W} : \sin \widehat{R_2 P_2},$$

or,
$$P_2 : W : : \sin (\alpha + \epsilon) : \cos (\theta - \epsilon) ;$$

$$\therefore \quad P_2 = W . \frac{\sin (\alpha + \epsilon)}{\cos (\theta - \epsilon)} = W . \frac{\sin \alpha + \mu \cos \alpha}{\cos \theta + \mu \sin \theta}.$$

122. If the power acts horizontally, $\theta = -\alpha$, and hence in this case

$$P = W . \frac{\sin (\alpha \mp \epsilon)}{\cos (\alpha \mp \epsilon)},$$

$$= W . \tan (\alpha \mp \epsilon) = W . \frac{\tan \alpha \mp \mu}{1 \pm \mu \tan \alpha}.$$

Again, if the power acts along the plane, $\theta = 0$, and hence in this case

$$P = W . \frac{\sin (\alpha \mp \epsilon)}{\cos \epsilon},$$

or,
$$P = W . (\sin \alpha \mp \mu \cos \alpha).$$

123. *To find the inclination of the plane down which a body is on the point of sliding by its own weight.*

Let α be the inclination of the plane, then, by the preceding, if W be on the point of sliding down the plane,

$$P = W (\sin \alpha - \mu \cos \alpha).$$

But, by the hypothesis, $P = 0$;

$$\therefore \quad \sin \alpha - \mu \cos \alpha = 0,$$

$$\therefore \quad \tan \alpha = \mu ;$$

that is, the inclination of the plane is the angle of repose. Hence, one of the methods which have been employed for determining experimentally the coefficient of friction is that of observing the inclination at which the one body is on the point of sliding over the other. The tangent of this angle gives the value of μ for surfaces that have been some time in contact.

124. *To show that the best angle of traction is the angle of repose.*

In Art. 121 it was shown, that if θ be the angle of traction, the power necessary to bring a body whose weight is W into a state bordering on a motion up a plane whose inclination is α, is

$$W\frac{\sin(\alpha+\epsilon)}{\cos(\theta-\epsilon)}.$$

It is required to find the value of θ that will make this the least possible. Since θ enters into the denominator only, the problem will be solved by finding the value of θ, which gives the denominator its greatest possible value. The greatest value of any cosine is unity, hence θ must be taken such that

$$\cos(\theta-\epsilon)=1;$$

$$\therefore \quad \theta-\epsilon=0,$$

$$\text{or,} \quad \theta=\epsilon.$$

125. *To find the limits of equilibrium in the screw.*

Let W be the weight sustained by the screw, h the distance between the threads, r the radius of the cylinder, and μ the coefficient of friction.

The weight sustained by the screw may, as already seen, be regarded as a weight sustained by a horizontal power upon an inclined plane, whose height is h and base $2\pi r$; that is, upon a plane, the tangent of whose inclination is $\frac{h}{2\pi r}$. Substituting this value for tan α, in the expression given in Art. 122,

$$P=W.\frac{h\mp 2\mu\pi r}{2\pi r\pm\mu h},$$

P being supposed to act at the surface of the cylinder. If the weight be sustained by a power P′ acting at the extremity of a lever whose length is R, then

$$P' = P \cdot \frac{r}{R};$$

$$P' = W \cdot \frac{r}{R} \cdot \frac{h \mp 2\mu\pi r}{2\pi r \pm \mu h}.$$

126. *To find the limits of the ratio of the power to the normal pressures in an isosceles wedge.*

Let ABC (fig. Art. 101) be the wedge, and let the angle ACB $= 2\alpha$. Let P, the power, act at H. Let the resistances on each side of the wedge $= \frac{1}{2}R'$, and act at the points D, E, in the directions DG, EG, making an angle ϵ (ϵ = angle of repose) with the normal, and therefore angle $90° - \epsilon$ with the sides of the wedge. Let the power be on the point of overcoming the resistance; then the angles ADG, BFG are equal to $90° - \epsilon$. Hence,

$$P : \tfrac{1}{2}R' : : \sin DGE : \sin DGH;$$

$$\therefore \quad P : \tfrac{1}{2}R' : : \sin\{180° - 2(\alpha + \epsilon)\} : \sin\{90° + (\alpha + \epsilon)\};$$

$$\therefore \quad P = \tfrac{1}{2}R' \frac{\sin 2(\alpha + \epsilon)}{\cos(\alpha + \epsilon)},$$

$$= R' . \sin(\alpha + \epsilon).$$

Secondly: Let the resistance be on the point of overcoming the power; then the angles CDG, CEG are each $= 90° - \epsilon$,

$$\therefore \quad P : \tfrac{1}{2}R' : : \sin\{180° - 2(\alpha - \epsilon)\} : \sin\{90° + (\alpha - \epsilon)\};$$

$$\therefore \quad P = \tfrac{1}{2}R' \frac{\sin 2(\alpha - \epsilon)}{\cos(\alpha - \epsilon)}.$$

$$= R' \sin(\alpha - \epsilon).$$

f $\frac{1}{2}R$ be the normal resistance, $R = R' \cos \epsilon$. Hence,

$$P = R . \frac{\sin(\alpha \pm \epsilon)}{\cos \epsilon},$$

r,

$$P = R(\sin \alpha \pm \mu \cos \alpha).$$

127. *To find the greatest angle for a wedge that shall be retained in a cleft by the force of friction alone.*

By the preceding, when the resistance is on the point of overcoming the power,

$$P = R\,(\sin\alpha - \mu\cos\alpha).$$

Therefore, when $P = 0$,

$$\sin\alpha - \mu\cos\alpha = 0,$$

or, $$\tan\alpha = \mu.$$

Hence α, or half the opening of the wedge, is equal to the angle of repose.

EXAMPLES.

1. A plane rises 5 in 13; find the greatest weight which can be sustained upon it by a power of 20 lbs. acting along the plane, the coefficient of friction being $\frac{1}{4}$. Ans. 130 lbs.

2. A horizontal force of 11 lbs. is on the point of drawing a weight of 13 lbs. up a plane whose inclination is equal to $\tan^{-1}(\frac{1}{3})$; determine the coefficient of friction. Ans. $\frac{2}{5}$.

3. The least power which will draw a weight of 205 lbs. up an inclined plane is 70 lbs., and the least power which will sustain the same weight is 20 lbs., the power in both cases acting along the plane; required the inclination of the plane, and the coefficient of friction. Ans. $\sin^{-1}(\frac{9}{41})$ and $\frac{1}{8}$.

4. A wheel 20 inches in radius, with an axle 6 inches in radius and weighing 70 lbs., turns upon a pivot 1 inch in radius. Th coefficient of friction is ·1; what power is required to raise a weigh of 100 lbs., the power acting downwards?* Ans. $31\frac{1}{199}$ lbs.

5. In the preceding what power is required if it act upwards? Ans. $30\frac{140}{201}$ lbs.

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* When an angle is small the value of the sine may be taken as equal that of the tangent.

6. The distance between the threads of a screw is such that a weight is just sustained without the action of any power; what is the least power which will overcome any weight W?

Ans. $\frac{2 \mu W}{1 - \mu^2}$.

7. The mechanical advantage of a screw press is 240, that of the screw alone is 20; determine the power which will just overcome a resistance of 260 lbs., the coefficient of friction being $\frac{1}{2}$.

Ans. $12\frac{2}{9}$ lbs.

8. Two weights, P and Q, rest upon a double inclined plane, and are connected by a cord passing over a pulley at the summit; the coefficient of friction is the same for both planes, and P is just on the point of descending; determine the coefficient of friction.

Let α and β be the inclinations of the planes, then the coefficient of friction is equal to

$$\frac{P \sin \alpha - Q \sin \beta}{P \cos \alpha + Q \cos \beta}.$$

9. The inclination of a plane is such that a body placed upon it is upon the point of sliding down by its own weight; show that the least power which will draw the body up the plane is double the power which sustains the same weight when the plane is smooth.

CHAPTER VII.

STATICAL PROBLEMS.

128. In attempting the solution of statical problems the student should remember,

i. That in the case of any number of forces, P_1, P_2, &c., acting upon a point, the equations of equilibrium are two in number, viz.:

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0,$$
$$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \&c. = 0,$$

where α_1, α_2, &c., are the angles made by the directions of the forces with any selected line.

ii. That in the case of forces acting at different points, but lying in the same plane, the equations of equilibrium are three in number, viz.: two as given above, and a third

$$P_1 p_1 + P_2 p_2 + \&c. = 0,$$

where p_1, p_2, &c., are the perpendiculars to the directions of the forces from any selected point in the plane of the forces.

iii. That in the case of parallel forces lying in the same plane the equations of equilibrium are two in number,

$$P_1 \quad + P_2 \quad + \&c. = 0,$$
$$P_1 p_1 + P_2 p_2 + \&c.,$$

where p_1, p_2 mean the same as above.

iv. That in the case of parallel forces not lying in the same plane the equations of equilibrium are three in number,

$$P_1 \quad + P_2 \quad + \&c. = 0,$$
$$P_1 h_1 + P_2 h_2 + \&c. = 0,$$
$$P_1 k_1 + P_2 k_2 + \&c. = 0,$$

here h_1, h_2, &c., and k_1, k_2, &c., are the perpendicular distances of e forces from any two planes of reference parallel to their direction.

129. If three forces, not parallel to each other, acting at any ints in a rigid body are in equilibrium, they will lie in the same ane. There must therefore be a point in which their directions, produced, will meet. For let the directions of any two be pro- ced till they meet, then their resultant must pass through this int, and since there is equilibrium, the third force must also s through this point. Consequently, the triangle of forces will ly to three forces in equilibrium, acting at any points in a rigid ly, and in many cases the relations between the forces may be re readily obtained by its aid than by the more general equa- ns given above.

The triangle of forces will give two of the three equations, and third may be obtained by taking the moments of any two of forces about any point in the direction of the third force; and ce the resultant of the two must be equal and opposite to the rd, these moments must be equal.

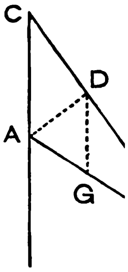
130. When more than three forces act at various points of a id body, many problems of equilibrium may be conveniently ved, if all the forces can be brought to act at any two points in body. For then, since there is equilibrium, the resultant of e set of forces must be equal and opposite to the resultant of other set, and consequently the resultant of each set must act ng the line joining the two points. And if at each point a ce equal and opposite to the resultant be introduced, equili- ium will subsist between each set of forces separately.

131. *One extremity of a beam* AB *is fastened by a pivot at* A, *d a cord of given length is tied to the other extremity* B, *and tened to a given point* C, *lying above* A *in the same vertical line, uired the tension in the string.*

Let W be the weight of the beam, acting at its centre of gravity

Let AB = l, AG = a, AC = h, and BC = c.

The three forces acting upon the beam are the weigl beam, the tension of the string, and the resistance of the pivot. Through G draw the vertical line GD. Join AD, and let T be the tension of the string. T acts along CB, and W along GD, consequently the resultant of T and W must pass through D. But since there is equilibrium, this resultant must be equal and opposite to the third force, viz., the resistance of the pivot, and consequently AD is the direction in which this resistance acts.



The sides of the triangle ACD are severally paralle directions of the three forces,

$$\therefore \qquad T : W :: CD : AC,$$

$$\text{or,} \qquad T = W\frac{CD}{h}.$$

But since GD and AC are parallel, CD : CB : : AG

$$CD : c :: a : l,$$

$$\therefore \qquad CD = \frac{ac}{l},$$

$$\text{and} \qquad T = W\frac{ac}{hl}.$$

132. As an exercise in the use of the general equations brium, find the complete solution of the preceding probl is, find the tension in the string, the pressure on the peg direction of this pressure.

Let T be the tension, R the pressure on the peg, and clination of R to the vertical line.

Then, resolving vertically and horizontally, and ta moments about A, we have

$$R \cos\theta + T \cos C - W = 0. \qquad \text{(i.)}$$

$$R \sin\theta - T \sin C = 0. \qquad \text{(ii.)}$$

$$Wa \sin A - Tl \sin B = 0. \qquad \text{(iii.)}$$

From (iii.) we obtain directly

$$\mathrm{T}=\frac{\mathrm{W}a\sin\mathrm{A}}{l\sin\mathrm{B}}=\frac{\mathrm{W}ac}{hl}.$$

Eliminating θ between i. and ii. we have

$$\mathrm{R}^2=\mathrm{T}^2+\mathrm{W}^2-2\mathrm{TW}\cos\mathrm{C};$$

$$=\frac{\mathrm{W}^2}{h^2l^2}(a^2c^2+h^2l^2-2achl\cos\mathrm{C});$$

$$=\frac{\mathrm{W}^2}{h^2l^2}\{a^2c^2+h^2l^2-al(h^2+c^2-l^2)\};$$

or,

$$\mathrm{R}=\frac{\mathrm{W}}{hl}\sqrt{al^3+(l-a)(lh^2-ac^2)};$$

Eliminating R between i. and ii. we have

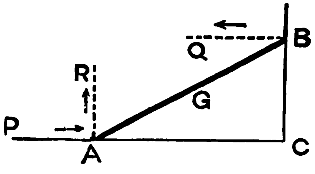
$$\cot\theta=\frac{\mathrm{W}-\mathrm{T}\cos\mathrm{C}}{\mathrm{T}\sin\mathrm{C}};$$

$$=\frac{hl-ac\cos\mathrm{C}}{ac\sin\mathrm{C}};$$

and since the three sides of the triangle ABC are given, cos C and sin C can thence be found.

133. One extremity B of a beam, AB, rests against a smooth vertical wall BC, and the other upon a smooth horizontal plane AC; required the horizontal force to be applied at A in order that the beam may rest in a given position; also the pressures on the wall and plane.

Let l be the length of the beam, W its weight, and G its centre of gravity. Let AG = A, and the angle BAC = α. Let P be the force required, R the resistance of the plane, and Q the resistance of the wall.



Taking the vertical and horizontal

directions as the lines of reference, and taking the moments [illegible] A, we have as the equations of equilibrium

$$R - W = 0.$$

$$P - Q = 0.$$

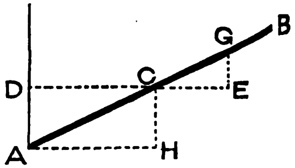
$$Ql \sin \alpha - Wa \cos \alpha = 0.$$

Hence, $$P = Q = \frac{Wa \cot \alpha}{l},$$

and $$R = W.$$

134. *A beam rests freely upon a peg, with one end against a smooth vertical wall; required the position of equilibrium, and the pressures on the wall and peg.*

Let AB be the beam resting upon the peg C and against the wall AD. Let G be the centre of gravity of the beam, and W its weight. Draw DE, AH, perpendicular to AD, and GE, CH, parallel to AD. Let GA = a, and CD = b. When there is equilibrium, let CA = x.



The resistance of the wall is equal and opposite to the pressure of the beam against it. Let R denote this resistance, and Q the resistance of the peg. The beam then is acted on by three forces; by R acting perpendicularly to the wall, by Q acting perpendicularly to the beam, and by W acting vertically. The sides of the triangle ACD are severally perpendicular to the directions of these forces, and therefore, since there is equilibrium, Q : W : : AC : CD, and R : W : : AD : CD;

$$\therefore \qquad Q = W\frac{x}{b}, \qquad \text{(i.)}$$

and $$R = W\frac{\sqrt{(x^2 - b^2)}}{b} \qquad \text{(ii.)}$$

In order that equilibrium may subsist, the resultant of R and W must pass through C; therefore

$$R \times CH = W \times CE.$$

it CH = AD = $\sqrt{(x^2 - b^2)}$, and since the triangles CEG, CAD are nilar,

$$CE : CG :: CD : AC,$$

or,

$$CE : a - x :: b : x.$$

Hence, $$R\sqrt{(x^2 - b^2)} = Wb\,.\,\frac{a-x}{x} \qquad \text{(iii.)}$$

The equations i. ii. and iii. are sufficient to determine the three nknown quantities R, Q, and x. For substituting in equation ii. the value of R given by equation ii. we have

$$x^2 - b^2 = b^2.\frac{a-x}{x};$$

$$\therefore \qquad x^3 = b^2a, \text{ or } x = b\sqrt[3]{\left(\frac{a}{b}\right)}.$$

ubstituting this value of x in equations i. and ii. we have

$$Q = W\sqrt[3]{\left(\frac{a}{b}\right)} \text{ and } R = W\left\{\left(\frac{a}{b}\right)^{\frac{2}{3}} - 1.\right\}^{\frac{1}{2}}$$

135. In the preceding example, if the beam, instead of resting eely upon the peg C, were fixed by a pivot at any point C, the nly condition of equilibrium is that the resultant of R and W ass through C. Therefore, if CA = c, the pressure on the wall is btained immediately from equation iii.; whence

$$R\sqrt{(c^2 - b^2)} = Wb\frac{a-c}{c}; \;\therefore\; R = W\frac{b}{c}.\frac{a-c}{\sqrt{(c^2-b^2)}}.$$

136. *One extremity* B *of a beam* AB (fig. Art. 133) *rests against vertical wall, and the other* A, *upon a horizontal plane; to find e position of the beam when on the point of slipping.*

Let l be the length of the beam, W its weight, G its centre of avity, and let AG be equal to a. Let R be the normal resistance A, and Q that at B. Let μ_1 be the coefficient of friction for e plane, and μ_2 for the wall; then the force of friction at A will a horizontal force μ_1R acting in the direction of P, and the force friction at B will be a vertical force μ_2Q acting upwards.

Let θ be the angle BAC when the beam is on the point of slipping. Then, taking the vertical and horizontal directions as the lines of reference, and taking the moments about A, we have

$$R+\mu_2 Q-W=0. \qquad \text{(i.)}$$

$$\mu_1 R-Q=0. \qquad \text{(ii.)}$$

$$\mu_2 Ql\cos\theta+Ql\sin\theta-Wa\cos\theta=0. \qquad \text{(iii.)}$$

From i. and ii. we obtain

$$R=\frac{W}{1+\mu_1\mu_2},$$

$$Q=\frac{\mu_1 W}{1+\mu_1\mu_2}.$$

Substituting these values in iii. we obtain

$$\tan\theta=\frac{a-\mu_1\mu_2(l-a)}{l\mu_1}.$$

137. *One extremity of a beam is fastened by a pivot, and the other rests upon a smooth horizontal plane; required the upward vertical pressure upon the pivot when the foot of the beam is acted on by a given horizontal force, the weight of the beam itself being neglected.*

Let the beam AB (see fig. Art. 133) be fastened by a pivot at B, and let a force P be applied horizontally at A. Let the angle BAC be equal to α.

The beam is in equilibrium from the force P, the resistance of the plane acting vertically at the point A and the resistance of the pivot. But when three forces are in equilibrium, any one must be opposite to the resultant of the other two; hence, the resistance of the pivot will act along the line BA. Let R′ denote the resistance of the pivot, then

$$R':P::\sin 90^\circ:\sin(90^\circ-\alpha);$$

$$\therefore \qquad R'=\frac{P}{\cos\alpha}.$$

The vertical pressure required is the vertical component of R′;

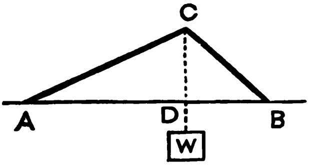
$$\therefore \qquad \text{vertical pressure on } B=R'\cos ABC;$$

$$=R'\sin\alpha;$$

$$=P\tan\alpha.$$

8. *Let* AC, BC *be two beams, of uniform thickness and density,* *g upon a horizontal plane at* A *and* B, *and fastened together at* *a pin; let* A *and* B *be connected by a cord; required the* *n in the cord when a weight* w *hangs from* C.

w_1 = the weight of the beam AC, and w_2 = the weight of the BC. Then, since the beams uniform thickness and density, y resolve w_1 acting at the centre ity of AC into two forces, each o $\frac{1}{2}w_1$ acting vertically down- at A and C; and, similarly,



y be resolved into two forces, each equal to $\frac{1}{2}w_2$ acting lly downwards at B and C. Hence, the whole force acting ards at C is $w + \frac{1}{2}(w_1 + w_2)$. Let $W = w + \frac{1}{2}(w_1 + w_2)$. Since s equilibrium, the upward vertical pressures upon C must W. Let T = the tension in the cord; let the angle CAB = the angle CBA = α_2.

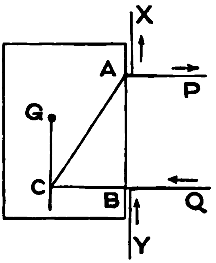
the preceding article, the vertical thrust of the beam AC is T tan α_1, and the vertical thrust of the beam BC is T

Hence,
$$T \tan \alpha_1 + T \tan \alpha_2 = W;$$

$$\therefore \quad T = \frac{W}{\tan \alpha_1 + \tan \alpha_2}.$$

. *To find the horizontal strain upon the hinges of a door.*

G be the centre of gravity of the door, and W its weight. e hinges be at A and B.



ead of the hinges, let the door be re- as supported by a vertical force X, and ontal force P acting at A, and a vertical and a horizontal force Q acting at B. ident that P and Q must act in oppo- ections.

GC be the vertical line drawn through raw BC perpendicular to GC. Let , and AB = b.

Remove the point of application of Y from B to A, of Q from B to C, and of W from G to C.

The resultant of W and Q at C must be equal and opposite to the resultant of P and X + Y at A. Each resultant must therefore act along AC. Let R represent the value of either; then

$$Q : W :: BC : AB,$$

or, $$Q : W :: a : b;$$

∴ $$Q = W\frac{a}{b},$$

also, $$Q : R :: BC : AC.$$

Again, since the forces at A are in equilibrium,

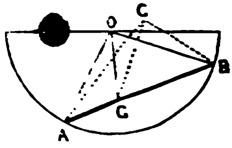
$$P : R :: BC : AC;$$

therefore, $$P = Q;$$

or the horizontal strain on each hinge is the same in amount, while opposite in direction.

140. *To find the position of equilibrium of a beam resting in a smooth hemispherical bowl.*

Let AB be the beam, and G its centre of gravity. Let $2l$ = the length of the beam, a = AG, and r = the radius of the bowl. The reactions of the bowl will act along the radii AO, BO. Consequently, in the position of equilibrium, the vertical line through O must pass through G. Let θ = the angle OGB, and let ϕ = the angle OAB = the angle OBA. Then



$$AG : AO :: \sin AOG : AGO,$$

or, $$a : r :: \sin(\theta - \phi) : \theta;$$

∴ $$\frac{\sin(\theta-\phi)}{\sin\theta} = \frac{a}{r};$$

∴ $$\cos\phi - \cot\theta \sin\phi = \frac{a}{r};$$

∴ $$\cot\theta = \frac{r\cos\phi - a}{r\sin\phi}.$$

But $r \cos\phi = l$, and therefore $r \sin\phi = \sqrt{(r^2 - l^2)}$;

$$\therefore \qquad \cot\theta = \frac{l-a}{\sqrt{(r^2-l^2)}}.$$

141. *To find the position of a beam resting in a rough hemispherical bowl when on the point of slipping.*

As in the preceding, let θ be the angle made by the beam with the vertical line through its centre of gravity, and ϕ the angle between the beam and the radius through either extremity. Let ϵ be the angle of repose.

Let AB (fig. Art. 140) be the beam in the position required, and let the reactions of the bowl act in the directions AC, BC. Then the angle $\text{CAB} = \phi - \epsilon$, and the angle $\text{CBA} = \phi + \epsilon$. As before, the vertical line through C must pass through G. Let CG be vertical; then, by hypothesis, the angle $\text{CGB} = \theta$.

$$\text{AG} : \text{GC} :: \sin \text{ACG} : \sin \text{CAG},$$

or,
$$a : \text{GC} :: \sin(\theta - \phi + \epsilon) : \sin(\phi - \epsilon).$$

And
$$\text{GC} : \text{GB} :: \sin \text{CBG} : \sin \text{GCB},$$

or,
$$\text{GC} : 2l - a :: \sin(\phi + \epsilon) : \sin(\theta + \phi + \epsilon).$$

Hence,
$$\frac{\sin(\theta - \phi + \epsilon)}{\sin(\theta + \phi + \epsilon)} \cdot \frac{\sin(\phi + \epsilon)}{\sin(\phi - \epsilon)} = \frac{a}{2l - a}.$$

$$\therefore \quad \frac{\sin\theta\cos(\phi - \epsilon) - \cos\theta\sin(\phi - \epsilon)}{\sin\theta\cos(\phi + \epsilon) + \cos\theta\sin(\phi + \epsilon)} \cdot \frac{\sin(\phi + \epsilon)}{\sin(\phi - \epsilon)} = \frac{a}{2l - a}.$$

Dividing numerator and denominator by $\sin\theta \sin(\phi - \epsilon) \sin(\phi + \epsilon)$,

$$\frac{\cot(\phi - \epsilon) - \cot\theta}{\cot(\phi + \epsilon) + \cot\theta} = \frac{a}{2l - a}.$$

Whence $\cot\theta = \frac{1}{2l}\{(2l - a)\cot(\phi - \epsilon) - a\cot(\phi + \epsilon)\}$.

The value of ϕ is known from the equation $r \cos\phi = l$, and ϵ is given, therefore θ is determined.

142. *A beam, leaning against a small cylindrical peg* B, *rests with one end* A *upon a horizontal plane; the coefficient of friction is the same between the beam and the peg, and between the beam and the plane: determine the position of the beam when on the point of slipping.*

Let a be the distance of the centre of gravity of the beam from A, θ the inclination of the beam to the horizontal plane, and b the height of the peg above the plane.

Let R_1 be the resistance of the plane, and R_2 the resistance of the peg, and ϵ the angle of repose; then,

$$R_1 \hat{} R_2 = \theta,$$

$$R_2 \hat{} W = 180^\circ - (\theta - \epsilon),$$

$$W \hat{} R_1 = 180^\circ - \epsilon.$$

Hence, by the triangle of forces,

$$R_1 = \frac{W \sin(\theta - \epsilon)}{\sin\theta},$$

and
$$R_2 = \frac{W \sin \epsilon}{\sin\theta},$$

Also, taking the moments of R_2 and W about A, we have

$$\frac{b \sin\epsilon . \cos\epsilon}{\sin^2\theta} = a \cos\theta;$$

and therefore
$$\sin^2\theta \cos\theta = \frac{b \sin\epsilon \cos\epsilon}{a},$$

or,
$$\cos^3\theta - \cos\theta + \frac{b \sin 2\epsilon}{2a} = 0.$$

143. A *bar* AB, *whose weight is* W, *rests with one end* A *upon a horizontal plane, a cord fastened to* B *passes over a fixed pulley placed above the plane and carries a weight* P; *determine the position of equilibrium, and the pressure upon the plane, when* A *is on the point of slipping towards the pulley, disregarding the friction of the pulley.*

Let θ be the inclination of the beam to the horizontal line, and ϕ that of the cord. Let R be the resistance of the plane, and ϵ the angle of repose; then,

$$\widehat{RW} = 180^\circ - \epsilon,$$

$$\widehat{WP} = 90^\circ + \phi,$$

$$\widehat{PR} = 90^\circ - (\phi - \epsilon).$$

Therefore, by the triangle of forces,

$$\frac{W}{P} = \frac{\cos(\phi - \epsilon)}{\sin \epsilon}. \qquad \text{(i.)}$$

$$\frac{R}{P} = \frac{\cos \phi}{\sin \epsilon}. \qquad \text{(ii.)}$$

Let l be the length of the bar, and a the distance of its centre of gravity from A; then, taking the moments about A,

$$Wa \cos \theta = Pl \sin(\phi - \theta);$$

and therefore

$$\tan \theta = \frac{Pl \sin \phi - Wa}{Pl \cos \phi}. \qquad \text{(iii.)}$$

The value of ϕ is determined by equation i., and thence the value of R is found from equation ii., and the value of θ from equation iii.

EXAMPLES.

1. Two cords bearing weights P and Q respectively, and passing over fixed pulleys, are attached to a weight W; show that in the position of equilibrium the angles made by the cords with the vertical line have for their cosines $\frac{W^2 + P^2 - Q^2}{2PW}$ and $\frac{W^2 + Q^2 - P^2}{2QW}$ respectively.

2. A beam, AB, of uniform thickness and density rests with its lower end A upon a smooth plane whose inclination is α, and its upper end B upon a smooth plane whose inclination is β; show that if θ be the inclination of the beam to the vertical line, $\cot \theta = \frac{1}{2}(\cot \alpha - \cot \beta)$.

3. A beam, AB, of uniform thickness and density rests with one end A upon a rough plane whose inclination is α, and the other end B against a smooth vertical wall; show that if θ be the inclination of the beam to the wall when on the point of slipping up the plane, $\tan\theta = 2\tan(\alpha+\epsilon)$, where ϵ is the angle of repose for the plane.

4. The lid of an open desk rests against a smooth vertical wall, required the magnitude and direction of the pressure on the hinge, the lid being supposed to be of uniform thickness and density.

Let W be the weight of the lid, $2a$ the width, and $2b$ the distance of the hinge from the wall, then the pressure on the hinge is $\frac{W}{2}\cdot\frac{\sqrt{(4a^2-3b^2)}}{\sqrt{(a^2-b^2)}}$; and if θ be the inclination of the pressure to the vertical line, $\tan\theta = \frac{b}{2\sqrt{(a^2-b^2)}}$.

5. Two weights, P and W, balance each other on a straight lever moveable about a cylindrical pivot, find the ratio of P to W when P is on the point of descending; the arms of the lever being a and b respectively, r the radius of the pivot, and μ the coefficient of friction. Ans. $\frac{b+\mu r}{a+\mu r}$

6. If a beam rest with one extremity A in a smooth hemispherical bowl, and with the other projecting beyond the edge, show that in the position of equilibrium,

$$\sin\theta = \frac{a+\sqrt{(a^2+32r^2)}}{8r},$$

θ being the inclination of the beam to the vertical line, r the radius of the bowl, and a the distance of A from the centre of gravity of the beam.

7. What must be the length of a beam in order that, when resting within a sphere, its limiting position may be vertical?

Let μ be the coefficient of friction, and r the radius of the sphere then the required length is equal to

$$\frac{2r}{\sqrt{(1+\mu^2)}}.$$

8. Find the position of equilibrium of a balance when unequal weights P, Q, are suspended from A and B, the extremities of the beam. Let $AB = 2a$, W the weight of the balance, b the distance of AB from the fulcrum, and h the distance of the centre of gravity of the balance from the fulcrum. Then, if θ be the angle the beam makes with the horizontal line when there is equilibrium,

$$\tan\theta = \frac{(P-Q)a}{(P+Q)b+Wh}.$$

9. One extremity A of a beam rests against a smooth hemispherical bowl, and the other against a smooth vertical plane passing through the centre of the bowl—the radius of the bowl being greater than the length of the beam—then, if θ be the inclination of the beam to the vertical plane when in equilibrium, l the length of the beam, a the distance of its centre of gravity from A, and r the radius of the bowl,

$$\sin\theta = \sqrt{\left(\frac{l^4 - a^2r^2}{l^4 - a^2l^2}\right)}.$$

10. The lower extremity A of a beam rests against a vertical wall, a cord attached to the beam at a point B, between the centre of gravity and A, is fastened to the wall at a point C vertically above A; show that if ϕ be the inclination of the cord to the wall, and θ that of the beam, the position of the beam when A is on the point of slipping down is determined by the following equations:—

$$a(\cot\phi + \mu) = b(\cot\theta - \mu),$$
$$c\sin\phi = b\sin\theta,$$

a being the distance of B from the centre of gravity, b the distance of B from A, and c the length of the cord.

11. In the preceding, if the wall be perfectly smooth, and $b = a$, then will $c = a$, and the beam will rest in any position so long as A is below C.

12. A beam AB rests upon two inclined planes whose inclinations are α_1 and α_2 respectively, show that if a be the distance of

the centre of gravity from A, b its distance from B, ϵ the angle
repose, and θ the inclination of the beam to the vertical line whe
A is on the point of slipping up,

$$\tan\theta = \frac{a+b}{b\cot(\alpha_2-\epsilon) - a\cot(\alpha_1+\epsilon)}.$$

13. A thin hemispherical bowl is mounted with a cylindric
rim formed of a substance whose specific gravity is four times th
of the substance of the bowl, what must be the height of the ri
that the bowl may rest upon a horizontal plane in any position?

The height of the rim is equal to half the radius of the bowl.

14. In the preceding, if the bowl be thick, having its extern
and internal radii equal to a and b respectively, show that t
height of the rim is equal to

$$\left(\frac{a^2+b^2}{8}\right)^{\frac{1}{2}}.$$

15. Four balls, of equal radii but unequal weights, rest with
a hollow sphere in such a way that their centres lie in the sa
vertical plane, and each ball is in contact with the sphere, find t
position of equilibrium.

Let W_1, W_2, W_3, W_4, be the weights of the balls. Let θ be t
inclination to the horizon of the line joining the centre of the fi
ball with the centre of the sphere, and let α be the angle subtend
at the centre of the sphere by the line joining the centres of a
two adjacent balls, then

$$\tan\theta = \frac{W_1 + W_2\cos\alpha + W_3\cos 2\alpha + W_4\cos 3\alpha}{W_2\sin\alpha + W_3\sin 2\alpha + W_4\sin 3\alpha}.$$

DYNAMICS.

CHAPTER VIII.

ON THE LAWS OF MOTION.

When forces not in equilibrium act upon a body, motion sue. The degree of rapidity with which the body moves d its velocity.

Velocity may be either uniform or variable. Velocity is when the body describes *all* equal spaces in equal times, therefore be measured by the space described in a unit of t is customary to take a second for the unit of time, and r the unit of measure. Thus, if a body moving with a velocity have passed through 1000 feet in 8 seconds, its is 1000 ÷ 8, or 125; and generally, if s be the space in ugh which a body has passed in t seconds, then if v denote city,

$$v = \frac{s}{t}, \text{ or } s = vt.$$

ity is variable when *all* equal spaces are *not* described in nes. When a body in motion has a variable velocity, its at any moment may be measured by the space which e described in a unit of time, if at that moment the were to cease to vary.

FIRST LAW OF MOTION. *A body in motion, not acted on by rnal force, will continue to move in a straight line, and with m velocity.*

This is equivalent to the assertion, that matter possesses no inherent power of changing the direction or the state of its motion.

The truth of this law must be decided by an appeal to experiment. The powers with which matter has been endowed can evidently be learnt only by observation. Every attempt to prove their existence by *à priori* demonstration will be found to assume in some way or other the property under consideration.

The first law of motion cannot, however, be established by any *direct* experiment, for the prescribed conditions can under no circumstances be fulfilled. No where can we find a body which is not acted on by some external force. Every particle of matter is subject to a variety of external influences. Thus, if a ball be rolled along the ground, it is acted on by the attraction of all surrounding matter, by the resistance of the atmosphere, and by the force of friction; and it moves neither in a straight line, nor with a uniform velocity. It is found, however, that the more we lessen the influence of external force, the more nearly does the motion become direct and uniform.

If a ball be rolled along a smooth pavement, it will move for a longer time and in a line more nearly straight than when thrown with the same velocity along a rough road; and still more so if rolled along a sheet of ice.

If a weight suspended by a thread from any point be made to oscillate, it will after a time come to rest. One of the external forces acting upon the body is the resistance of the air. If this be diminished by causing the body to move within the exhausted receiver of an air pump, the oscillation will continue for a longer period; and the more perfect the vacuum the longer will the motion continue.

On a railway, after a train has acquired the desired velocity, it is no longer necessary for the engine to work, except so far as to overcome the effects of external forces, such as the friction of the rails and the resistance of the atmosphere.

The most convincing evidence of the truth of this law is found in the accordance of the consequences deducible from it with observed phenomena. It is impossible to doubt the correctness of a principle, upon the assumption of which the motions of the moon can be predicted with almost unerring certainty, and the time of an eclipse foretold within the fraction of a second.

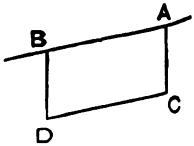
147. It follows from the first law of motion, that if a force continue to act upon a body already in motion, its velocity will continually change. This change may be either uniform or variable; uniform if the force be constant, variable if the force vary during the time of its action.

The change in the velocity resulting from the action of any force is uniform, if in *all* equal periods of time there be an equal increase or an equal diminution in the velocity. The body is then said to be *uniformly accelerated*, and the acceleration is measured by the velocity added (or subtracted) in a unit of time. Thus, if a body move with a uniform acceleration of 32, it gains or loses in each second of time a velocity of 32 feet per second; that is, in one second it gains or loses a velocity of 32, in two seconds a velocity of 64, in three seconds a velocity of 96, and so on. And hence, generally, if f represent the acceleration, and v be the velocity gained or lost in t seconds,

$$v = ft.$$

148. Forces which act during an extremely short time only, as, for example, when a body is set in motion by a sudden blow, or a ball is fired from a cannon, are termed *impulsive forces.* It will be seen hereafter that although, under the action of such a force, it sometimes appears that motion is communicated *instantaneously*, yet this is not really the case. The force, in fact, acts through a finite though short interval of time, and throughout that period the velocity of the body undergoes continual change. These successive changes, however, take place with such immense rapidity, that they cannot be brought under examination. We concern ourselves therefore only with the velocity and direction of the motion which takes place after the force has ceased.

149. Second Law of Motion. *When a force acts upon a body in motion, the change of motion produced is the same, both in magnitude and direction, as if the force acted on the body at rest.*



Thus, if a body move along the line AB with such a velocity that it would describe the space AB in one second; and if, when it arrives at A, a force act upon it such as of itself to cause a body to pass from A to C in one second, then at the end of the second the body will be found at D; the change of motion represented by BD being the same in magnitude and direction as if the force had acted upon the body when at rest. Each force produces its full effect in its own direction.

This law is proved by such experiments as the following:—

If a ball, lying on the deck of a vessel moving steadily through the water at a uniform rate, be struck with a blow of any magnitude and direction, its motion along the deck is the same as if the vessel had been at rest.

If a stone be dropped from the top of the mast, when a vessel is moving uniformly in any direction, and with any velocity, it will fall at the foot of the mast, just as it would if the vessel had been at rest.

If from any point a ball be let fall, and another ball be at the same instant projected forward horizontally with any velocity whatever, both balls will strike the ground at the same time. Here the ball at rest and the ball in motion are acted upon by the same vertical force; namely, the force of gravity, and both are caused to pass through the same vertical space in the same time.

If a person in a railway carriage throw a ball vertically upwards, it will not fall towards the back of the carriage, but will drop into the hands of the individual who projected it.

150. The force acting upon the moving body, in these and similar experiments, is either an impulsive force, or the force of gravity. Within the limits taken in the experiments, the force of gravity does not sensibly vary either in magnitude or direction.

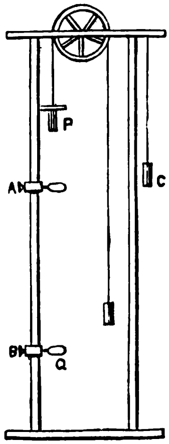
It is, therefore, only when a force acts for an indefinitely short time; or, if acting throughout a measurable interval, is uniform both in magnitude and direction, that the second law of motion can be regarded as established by such experiments as those described above. It is important that the student give particular attention to this, inasmuch as the general terms in which it has become customary to enunciate this law might otherwise lead him into error.

151. **The Third Law of Motion.** *When force communicates motion to a body, the acceleration varies directly as the force and inversely as the mass.*

Thus, if a pressure of 12 lbs. communicates motion to a mass weighing 20 lbs., and a pressure of 8 lbs. communicates motion to a mass weighing 10 lbs., the third law of motion declares that the acceleration when 12 lbs. moves 20 lbs. : the acceleration when 8 lbs. moves 10 lbs. : : $\frac{12}{20} : \frac{8}{10}$.

This law is verified experimentally by means of Attwood's machine, as explained in the following article.

152. **Attwood's Machine.** This consists of two pillars, one of which is graduated, supporting a pulley arranged so as to work with as little friction as possible. An open ring A and a stage B slide along the graduated pillar, and by means of screws can be fixed at any part of it. P and Q are two equal cylindrical weights, connected by a cord passing over the pulley. C is a pendulum beating seconds.



P and Q, being equal, will have of themselves no tendency to motion; but if a small bar be placed upon P, P will descend with an accelerated velocity, until it reach the ring A, which, allowing P to pass through, but intercepting the bar, removes the cause of acceleration. P will then, in accordance with the first law of motion, move uniformly with the velocity it had acquired on reaching A; and if the stage B be so placed that P may strike it exactly one second after reaching A, the distance AB will measure the velocity generated during the interval the bar was resting on P. If then the bar be allowed to rest on P for exactly one second before reaching A, we obtain the means of determining the velocity generated in one second; that is, the acceleration.

The entire mass moved consists, it will be seen, of the bar and the two weights; the force causing motion is the weight of the bar only.

If now we arrange for a series of experiments in which the combined weights of P, Q, and the bar, shall remain the same, but the weight of the bar vary, we shall find that the distance AB, which measures the velocity generated in one second, will vary in the same proportion as the weight of the bar. Thus if, for instance, in the first experiment P and Q are each $2\frac{3}{4}$ oz. and the bar $\frac{1}{2}$ oz., and in the second P and Q are each $2\frac{1}{2}$ oz. and the bar 1 oz., we shall find that AB in the latter case is twice as great as in the former. In both experiments the mass moved is the same, namely, 6 oz., but the pressure in the latter case is double the pressure in the former.

If, again, we arrange for another series of experiments in which the bar is the same, but the weight of P and Q varies, we shall find that the distance AB varies inversely as the entire weight moved. Thus, let the bar weigh 1 oz., and in the first case let P and Q weigh $5\frac{1}{2}$ oz. each, and in the second $2\frac{1}{2}$ oz. each, the space AB will in the latter case be twice as great as the former. In both experiments the force causing motion is the same, namely, 1 oz., but the mass moved in the latter case is $2\frac{1}{2}+2\frac{1}{2}+1$, or 6 oz., or only one half of that moved in the former, which is $5\frac{1}{2}+5\frac{1}{2}+1$, or 12 oz.

153. Momentum. The product of the mass of any body and the velocity with which it is moving is termed its *momentum*.

If P_1 and P_2 be forces acting upon bodies whose masses are M_1

and M_2, and if f_1, f_2 be the accelerations produced, then by the third law of motion

$$f_1 : f_2 :: \frac{P_1}{M_1} : \frac{P_2}{M_2}; \qquad \text{(i.)}$$

$$\text{or} \quad M_1 f_1 : M_2 f_2 :: P_1 : P_2. \qquad \text{(ii.)}$$

Now if v_1 and v_2 be the velocities generated in the bodies by the forces in the time t, then $v_1 = f_1 t$, and $v_2 = f_2 t$. And since by the proportion just given

$$M_1 f_1 t : M_2 f_2 t :: P_1 : P_2,$$

it follows that

$$M_1 v_1 : M_2 v_2 :: P_1 : P_2; \qquad \text{(iii.)}$$

or the momenta generated in equal times are as the forces producing them.

Hence the momenta generated in equal times by equal forces are equal, whether the masses be equal or unequal.

154. If the acceleration produced by gravity, or when a body alls freely, be known, then the third law of motion enables us to ind the acceleration when any mass is set in motion by any force. Let g denote the acceleration produced by gravity, and let f denote he acceleration when a force P moves a mass M, both P and M being measured by weight. Then by hypothesis g is the acceleraion produced in the same mass M by a force M. And since the masses are equal, the accelerations are as the forces, and therefore

$$f : g :: P : M;$$

r $\dfrac{\text{acceler. produced by any force}}{\text{acceler. produced by gravity}} = \dfrac{\text{force measured by weight}}{\text{mass measured by weight}}.$

155. One method of determining the value of g is supplied by he equation just established. For by means of Attwood's machine we can find the value of f experimentally for any values of P and M we please; and substituting these in the equation given above, e thence obtain the value of g.

Suppose, for instance, that when the bar is 1 oz., and the weights oz. each, we find the distance AB to be 2 feet, then in this case the acceleration is 2 feet, the force 1 oz., and the mass 1 $1 + 7\frac{1}{2} + 7\frac{1}{2}$, or 16 oz. Hence

$$\frac{2}{g} = \frac{1}{16}; \qquad \text{or } g = 32.$$

By repeating the experiment with different weights, taking the average of all the results, a more reliable va may be obtained than if we depended upon a single exper

The imperfections of Attwood's machine, arising from tion of the wheels and other causes, do not enable us to than to obtain a rough approximation to the value of g the way just described. Much greater accuracy is att means of experiments with pendulums. If t denote the t vibration measured in seconds, l the length of the pendul force of gravity, and π the ratio of the circumference of a its diameter, then, as is shown in Art. 240,

$$t = \pi\sqrt{\left(\frac{l}{g}\right)};$$

whence it follows that $$g = \frac{l\pi^2}{t^2}.$$

The time of a single vibration of a pendulum can be de with great accuracy by the simple device of observing the of vibrations made in a considerable period, and then div whole time by the number of vibrations. The length of th lum is found by measurement. By substituting these val obtained in the equation just given, and by taking the a a large number of results, it is found that at the sea lev latitude of London the value of g is 32·1908, or 32·2 nea

156. UNIT OF FORCE. The unit of force is the force a unit of time will generate in a unit of mass a unit of v

The unit of time is one second, and the unit of ma quantity of matter contained in a unit of weight.

In England the unit of weight is one pound, and th space is one foot. Hence the unit of force is the force one second will generate a velocity of one foot per second in a mass whose weight is one pound. If P denote this unit, then its value may be found from the equation (Art. 154),

$$\frac{1}{g}=\frac{\text{P}}{1\text{ lb.}}\text{ or P}=\frac{1\text{ lb.}}{g};$$

or the unit of force is very nearly the force exerted by a weight of half an ounce.

In France the unit of weight is one gramme, and the unit of space is one centimetre; and hence the unit of force is the force which in one second will generate a velocity of one centimetre per second in a mass whose weight is one gramme. This unit is termed a *dyne.* The value of g in centimetres is 981, and hence a dyne is $\frac{1}{981}$ of a gramme.

A convenient mode of designating these units of force is by the initial letters of the respective units of space, mass, and time. Thus the English unit may be designated the F.P.S. (foot—pound—second) unit; and the dyne, or French unit, the C.G.S. (centimetre—gramme—second) unit.

It will be seen from the above that using the F.P.S. unit, g or 32·2 is the number of such units exerted by a weight of 1 lb.; and hence if M denote the number of pounds in any mass, Mg is the number of units of force exerted by gravity upon that mass; that is, is the weight of the body in F.P.S. units.

Similarly, if we use the C.G.S. unit, g or 981 is the number of such units exerted by a weight of 1 gramme; and hence if M denote the number of grammes in any mass, Mg is as before the number of units of force exerted by gravity upon that mass; that is, is the weight of the body in dynes.

The student should bear in mind that the English pound is 16 thousandths of the weight, in the latitude of London, of a cubic foot of water at the standard density. The English unit of mass is therefore the quantity of matter contained in 16 thousandths of a cubic foot of water at the standard density. The gramme is the weight, in the latitude of Paris, of a cubic centimetre of water at its greatest density. The French unit of mass is therefore the quantity of matter contained in a cubic centimetre of wat at its greatest density.

It should also be noticed, that if M represent the number units of mass in any body, Mg will be the weight of the bo measured in units of force, whatever the value of g may be.

157. It has been seen (Art. 154) that if an acceleration f h been given to a body whose weight is W lbs., and if P be t force producing the acceleration,

$$P = \frac{Wf}{g} \text{ lbs.}$$

But if M be the number of units of mass in the body, W lbs = units of force. Hence $P = Mf$ units of force.

EXAMPLES.

1. What acceleration will be given to a mass weighing 10 by a force of 10 oz.? Answer, about 2 ft. per sec

2. What force will give to a mass of 1 ton an acceleration 3 feet per second? Answer 208·7 lbs. nearly

3. To what mass will a force of 100 lbs. give an acceleratio 161 feet per second? Answer 20 lb

4. If a force of m lbs. give to a mass of n lbs. an accelerat a, and a force of n lbs. give to a mass of m lbs. an acceleratio show that g is a mean proportional to a and b.

5. Show that to a mass of 32·2 lbs. the number of feet second in the acceleration given by any force is equal to the num of pounds in the force.

6. If a weight of W lbs. be placed on a table, and the t descend with an acceleration f, what is the pressure of the wei on the table? Answer $W\left(1 - \frac{f}{g}\right.$

7. If a weight of W lbs. be placed on a table, and the table scend with an acceleration f, what is the pressure of the weight n the table ?

Answer $W\left(1 + \frac{f}{g}\right)$.

8. What force acting for 11 minutes will give to a railway cariage weighing 15 tons a velocity of $24\frac{3}{20}$ miles per hour ?

Answer 56 lbs.

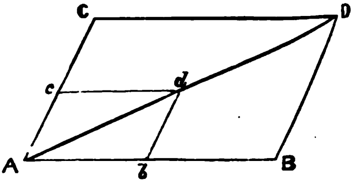
9. Three forces in geometric progression acting upon different nasses generate equal velocities in times which are in geometric rogression ; show that the masses also are in geometric progression.

10. Two forces generate equal velocities in times which are nversely as the masses, show that the forces are as the squares f the masses.

CHAPTER IX.

ON THE COMPOSITION AND RESOLUTION OF VELOCITIES.

158. *If* two uniform velocities *be simultaneously impressed upon a body, the resultant motion is* uniform and in a straight line; *and if two sides of a parallelogram represent the impressed velocities in magnitude and direction, the diagonal will represent the magnitude and direction of the resultant velocity.*



Let a body move uniformly along AB, with a velocity which would carry it from A to B in the time T. When the body is at A, let a velocity be impressed upon it such as of itself would cause the body to move uniformly from A to C in the time T; then, drawing the parallelogram ABDC, by the second law of motion, D is the position of the body at the end of the time T.

The motion of the body is along the straight line AD; for let t denote any time whatever, and take Ab so that

$$Ab : AB :: t : T;$$

then, since the velocity in the direction AB is uniform, Ab is the distance over which the body would move along AB in the time t.

Complete the parallelogram $Abdc$, then, by similarity of triangles,

$$bd : BD :: Ab : AB.$$

Therefore, $Ac : AC :: Ab : AB$,

or, $Ac : AC :: t : T$.

Therefore Ac is the distance over which the body would move along AC in the time t; and hence, by the second law of motion, the body will be found at d, a point in the line AD. And this is true for all values of t; therefore the body will always be found in the line AD.

Again, the motion along AD is uniform; for

$$\mathrm{A}d : \mathrm{AD} :: \mathrm{A}b : \mathrm{AB};$$

or,

$$\mathrm{A}d : \mathrm{AD} :: t : \mathrm{T};$$

or the space described is proportional to the time; that is, the motion is uniform.

Also, if AB, AC represent the spaces over which the impressed velocities would carry the body in one second, then AD is the space actually described by the body in one second; that is, the diagonal represents the magnitude of the resultant velocity.

159. *Conversely;* if any uniform velocity be represented by the line AD, it may be resolved into two velocities, in the direction AB and AC, whose magnitudes are represented by the lines AB and AC. For if any other lengths than AB and AC be taken to represent the component velocities, the resultant velocity will not be represented by AD.

160. It follows, from Art. 158, that if any number of uniform velocities be simultaneously impressed upon a body, the resultant motion will be uniform and in a straight line. The magnitude and direction of the resultant motion will be found by first taking the resultant of any two, then of this resultant and a third, and so on, until all the velocities have been compounded into one.

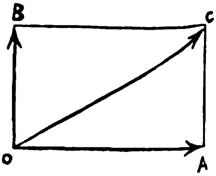
161. *If* any two velocities, *whether uniform or variable, be simultaneously impressed upon a body, the actual velocity of the body* at that instant *will be represented in magnitude and direction by the diagonal of the parallelogram whose sides represent the impressed velocities.*

This has been already proved, when both of the impressed velocities are uniform.

If one or both are variable, then let the lines AB and AC (fig. Art. 158) represent the velocities with which the bodies would have moved, if at the instant of reaching A all acceleration of the velocity were to cease. These then being uniform velocities, the line AD will represent, upon the same hypothesis, the actual velocity of the body at the same instant; that is, if all acceleration were to cease, the body would move in the direction AD with a velocity represented in magnitude by the line AD; and when a body moves with a variable velocity, the velocity at any instant is measured, both in magnitude and direction, by the path the body would describe in the succeeding second, if at that instant all variation, both in magnitude and direction, were to cease.

162. Hence, *if the velocity of a moving body at any instant be represented by any straight line in magnitude and direction, it may be resolved into two other velocities, represented in magnitude and direction by the sides of any parallelogram which has the given line for its diagonal.*

The most frequent application of this proposition is in the case in which the component velocities act in directions at right angles to each other.



Thus, let two velocities V_1 and V_2, simultaneously impressed upon a body at O, in directions at right angles to each other, be represented by the lines OA and OB; then, if v be the resultant velocity, v is represented by the line OC. But since $OC^2 = OA^2 + OB^2$,

$$v^2 = V_1^2 + V_2^2;$$

and if the angle COA be α,

$$\tan \alpha = \frac{V_2}{V_1}.$$

Again, if OC represent any velocity v, and it be required to resolve this velocity into two velocities, acting in directions at right angles to each other, namely, in the directions OA and OB; then if V_1 and V_2 be these components,

$$V_1 = v \cos \alpha,$$
$$V_2 = v \sin \alpha.$$

163. *If two* uniformly accelerated velocities *be simultaneously impressed upon a body, the resultant motion will be* in a straight line, and with a uniformly accelerated velocity; *and if the sides of a parallelogram represent the magnitudes and directions of the impressed velocities, the diagonal will represent the magnitude and direction of the resultant velocity.*

Let two velocities be impressed upon a body at A (fig. Art. 158), one of which would, if acting alone, cause the body to move from A to B in T seconds, with a velocity whose acceleration is f_1 feet per second; and the other of which would, if acting alone, cause the body to move from A to C in T seconds, with a velocity whose acceleration is f_2 feet per second; then, by the second law of motion, the place of the body at the end of T seconds is at D.

But since the spaces described in equal times by bodies moving with uniformly accelerated velocities, that is, under the action of constant forces, are in proportion to the accelerations

$$\mathrm{AB} : \mathrm{AC} : : f_1 : f_2.$$

Let Ab be the space which would be described along AB in any time t, and complete the parallelogram Abdc; then,

$$\mathrm{A}b : \mathrm{A}c : : \mathrm{AB} : \mathrm{AC},$$

whence, $$\mathrm{A}b : \mathrm{A}c : : f_1 : f_2;$$

and therefore Ac is the space which would be described in the direction AC in the time t. Consequently, by the second law of motion, the place of the body at the end of any time t is at d; that is, is in the line AD. The motion of the body is therefore in the straight line AD.

The resultant velocity is also uniformly accelerated; for since

$$\mathrm{A}d : \mathrm{A}b : : \mathrm{AD} : \mathrm{AB}.$$

Ad, the space described by the body in any time t, varies in the ame way as Ab, the space described in the same time by a body moving with a uniformly accelerated velocity.

Also, if AB and AC represent the impressed velocities f_1 and f_2, AD shall represent the resultant velocity. Let f be the resultant ccelerations; then, since AD is the space described under the action

of f, in the same time as the space AB would be described under t action of f_1; therefore, $AD : AB :: f : f_1$,
or AD represents f as AB represents f_1.

164. It follows from the preceding, that two uniform accele tions, acting simultaneously upon a body, may be compounded i a single acceleration, represented in magnitude and direction the diagonal of the parallelogram whose sides represent in mag tude and direction the component accelerations. And, converse a single acceleration, represented by the diagonal of a parallelogr may be resolved into two accelerations, represented by the sides the parallelogram.

165. *Def.* If a force act upon a body, and from any cause t motion of the body be not in the direction of the force, then t force which at any instant would, if it acted in the direction of t motion, produce the actual change which takes place, is termed t *effective force* at that instant. The force which really acts upon body is distinguished as the *impressed* force.

166. *To find the effective acceleration when a uniform force a upon a body constrained to move along a given line.*

Let f be the acceleration due to the impressed force, let ((fig. Art. 162) represent f in magnitude and direction, and let t body be constrained to move along OA; then the acceleration may be resolved into accelerations represented by OA and OB. I the conditions of the problem, all motion in the direction of C is prevented, the latter component may therefore be disregarde The other component OA acts in the direction of the motion, a is, therefore, the acceleration required. Hence,

$$\text{the effective acceleration} : f :: OA : OC;$$

or,
$$\text{effective acceleration} = f \cdot \frac{OA}{OC}.$$

If α be the angle between the direction of the force and the li of motion, then
$$\text{effective acceleration} = f \cdot \cos \alpha.$$

EXAMPLES.

1. A body moves in a direction inclined to the vertical line, at an angle of 60°, with a velocity of 100 feet; required the horizontal and vertical velocities.

Ans., Hor. vel. $= 50\surd 3$ ft.; vert. vel. $= 50$ ft.

2. A moving body has, at a given instant, a vertical velocity of 28 feet, and a horizontal velocity of 96 feet; required its actual velocity at that instant. Ans. 100 ft.

3. A moving body has, at a given instant, a vertical velocity of 86·6 feet, and a horizontal velocity of 50 feet; required the direction of its motion at that instant.

Ans. The body is moving in a direction inclined to the vertical line at an angle of 30° nearly.

4. Three velocities, each equal to V, are simultaneously impressed upon a body, one vertically, another horizontally, and the third in a direction inclined to the vertical line at an angle of 30°; required the actual velocity of the body. Ans. $V\surd(4+\surd 3)$.

5. Upon a body placed at A, one of the corners of a regular hexagon ABCDEF, velocities are simultaneously impressed such as would separately carry the body at a uniform rate to the other corners in three seconds; in what time will the body arrive at D?

Ans. 1 sec.

6. In the preceding, if each of the impressed velocities had been equal to V, what would have been the magnitude of the resultant velocity? Ans. $V(2+\surd 3)$.

CHAPTER X.

ON UNIFORM FORCES AND GRAVITY.

167. *To find the space described in* t *seconds, when a body moves from rest under the action of a uniform force.*

Let f denote the acceleration, and s the space described. Let the time be divided into n equal periods, each of which will consequently be equal to $\frac{t}{n}$. By Art. 147, the velocity at the end of the first of these periods will be $\frac{ft}{n}$, at the end of the second $\frac{2ft}{n}$, at the end of the third $\frac{3ft}{n}$, and so on.

If during these intervals the body be supposed to move uniformly with the velocity acquired at the end of each, then (since space = vel. × time, when velocity is uniform) the space described in the first interval will be $\frac{ft^2}{n^2}$, in the second $\frac{2ft^2}{n^2}$, and so on; the sum of this series will be the whole space described. Therefore,

$$
\begin{aligned}
s &= \frac{ft^2}{n^2} + \frac{2ft^2}{n^2} + \frac{3ft^2}{n^2} + \&\text{c.} \ldots\ldots + \frac{nft^2}{n^2} \\
&= \frac{ft^2}{n^2}(1 + 2 + 3 + \&\text{c.} \ldots\ldots + n) \\
&= \frac{ft^2}{n^2} \cdot \frac{n(n+1)}{2} \\
&= \frac{ft^2}{2} + \frac{ft^2}{2n}.
\end{aligned}
$$

The error arising from supposing the velocity to be uniform through each of these intervals will diminish, if the length of the intervals be diminished; that is, if n increase; and the more n increases, the more nearly shall we approach to the actual space described. As n increases, the value of $\frac{ft^2}{2n}$ diminishes without limit; and, therefore, $s = \frac{1}{2}ft^2$.

168. By the preceding, the space described in n seconds is $\frac{1}{2}fn^2$, and the space in $n - 1$ seconds is $\frac{1}{2}f(n-1)^2$; the difference between these is the space described in the nth second; therefore,

$$\text{space in } n\text{th second} = \tfrac{1}{2}f\{n^2 - (n-1)^2\},$$
$$= \tfrac{1}{2}f(2n-1).$$

But $2n - 1$ is the nth odd number, therefore the space described in the nth second equals $\frac{1}{2}f$ multiplied by the nth odd number.

169. *To find the relation between the space described and the velocity acquired by a body moving from rest under the action of a uniform force.*

By Art. 147, $v = ft$, and by Art. 167, $s = \frac{1}{2}ft^2$. Squaring both sides of the former, and multiplying both sides of the latter by $2f$, we obtain

$$v^2 = f^2t^2,$$
and
$$2fs = f^2t^2,$$
$$\therefore \quad v^2 = 2fs.$$

170. The three equations obtained in Articles 147, 167, and 169; namely, $v = ft$, $s = \frac{1}{2}ft^2$, and $v^2 = 2fs$, express the relation between ach pair of the three quantities; viz., the space described, the elocity acquired, and the time when a body moves from rest under he action of a uniform force. If any one of these quantities be nown, either of the other two may, by means of one or other of hese equations, be found directly. When gravity is the force onsidered, these equations are written $v = gt$, $s = \frac{1}{2}gt^2$, and $v^2 = 2gs$.

171. The following are examples of the application of these ormulæ to cases in which a body falls freely under the action of ravity.

Ex. 1. A stone falls from rest under the action of gravity; find he space described in 5 seconds.

In the formula, $s = \frac{1}{2}gt^2$, make $t = 5$; then

$$s = 16{\cdot}1 \times 25 \quad = 402{\cdot}5 \text{ feet}.$$

Ex. 2. Find the velocity which a stone will acquire by fal through 1610 feet.

In the formula, $v^2 = 2gs$, make $s = 1610$; then

$$v^2 = 64{\cdot}4 \times 1610,$$
$$= 103684;$$
$$\therefore \quad v = \sqrt{103684} = 322.$$

Ex. 3. How long must a body fall under the action of gr to acquire a velocity of 128·8 feet per second?

In the formula, $v = gt$, make $v = 128{\cdot}8$; then

$$128{\cdot}8 = 32{\cdot}2 \times t;$$
$$\therefore \quad t = 128{\cdot}8 \div 32{\cdot}2 = 4 \text{ seconds}.$$

Ex. 4. How far must a body fall under the action of gravi acquire a velocity of 96·6?

In the formula, $v^2 = 2gs$, make $v = 96{\cdot}6$; then

$$(96{\cdot}6)^2 = 64{\cdot}4 \times s;$$
$$\therefore \quad s = 144{\cdot}9.$$

172. *If the space described under the action of a uniform be divided into any number of parts, the square of the tin describing from rest the whole space is equal to the sum o squares of the times of describing from rest the several parts.*

Let the space described in the time T, under the action o force, be $a_1 + a_2 + \&c. \dots + a_n$; then, since $s = \frac{1}{2}ft^2$,

$$T^2 = \frac{2}{f}(a_1 + a_2 + \&c. \dots + a_n).$$

But if t_1, t_2, &c., be the times of describing from rest the tance a_1, a_2, &c., then

$$t_1^2 = \frac{2a_1}{f},$$
$$t_2^2 = \frac{2a_2}{f},$$
$$t_n^2 = \frac{2a_n}{f}.$$

Substituting these in the first equation, we have

$$T^2 = t_1^2 + t_2^2 + \&c. \dots + t_n^2.$$

COR. Hence, if T be the time of describing the whole distance, and t_1 the time of describing from rest any part of the distance, then, if t_2 be the time of describing from rest the remaining part,

$$t_2^2 = \mathrm{T}^2 - t_1^2.$$

Ex. 1. A stone can fall from a certain height to the ground in 10 seconds; if it be arrested at the end of 8 seconds, and afterwards let fall, how long will it be in descending the remaining distance?

Let t be the time required, then, by the proposition just demonstrated,

$$\begin{aligned} t^2 &= 10^2 - 8^2, \\ &= 36, \end{aligned}$$

or, $$t = 6.$$

Ex. 2. A stone falls to the ground in 6 seconds; how long was it in passing over the third tenth of the entire distance?

Let t be the time of describing the first tenth, then by the proposition given above,

$$10t^2 = 36,$$

or, $$t = \frac{3\sqrt{10}}{5}.$$

Hence, the time of describing three-tenths of the distance is

$$\frac{3\sqrt{10}\sqrt{3}}{5},$$

and the time of describing two-tenths is

$$\frac{3\sqrt{10}\sqrt{2}}{5};$$

therefore the time of describing the third tenth is

$$\tfrac{3}{5}\,(\sqrt{30} - \sqrt{20}).$$

173. *To find the tension in the string and the acceleration when a weight* P, *hanging freely, draws a weight* Q *along a smooth horizontal plane.*

Let T be the tension in the string, and f the acceleration.

The pressure causing motion in P is P − T; therefore, by Art. 154,

$$\frac{f}{g} = \frac{\mathrm{P} - \mathrm{T}}{\mathrm{P}}.$$

The pressure causing motion in Q is T; therefore, a

$$\frac{f}{g}=\frac{T}{Q}.$$

Solving these equations, we obtain

$$T=\frac{PQ}{P+Q},$$

and

$$f=g\cdot\frac{P}{P+Q}.$$

174. *To find the acceleration and the tension in the strin a weight* P *draws a weight* Q *over a fixed pulley, neglecti inertia of the pulley.*

Let T be the tension in the string, and f the acceleration.

The pressure causing motion in P is P − T; therefo Art. 154,

$$\frac{f}{g}=\frac{P-T}{P}.$$

The pressure causing motion in Q is T − Q; therefore

$$\frac{f}{g}=\frac{T-Q}{Q}.$$

Solving these equations, we obtain

$$T=\frac{2PQ}{P+Q}.$$

and hence,

$$f=g\cdot\frac{P-Q}{P+Q}.$$

175. The following are examples in illustration of the pre

Ex. 1. A weight of 9 ounces draws a weight of 7 ounces fixed pulley; find the space described from rest in t se neglecting the inertia of the pulley.

By Art. 174, the acceleration $=\frac{g(9-7)}{9+7}=4$ nearly.

Substituting this in the general formula $s=\frac{1}{2}ft^2$, we have

$$s=2t^2.$$

Hence, in one second the space described is 2 feet, in two seconds 8 feet, in three seconds 18 feet, and so on.

Ex. 2. A weight of 9 lbs. is drawn along a smooth horizontal table by a weight of 1 lb. hanging vertically by a string passing over a pulley at the edge of the table; find the space described from rest in 3 seconds, the velocity acquired in 4 seconds, and the space described in the 5th second.

By Art. 173, the acceleration $= \frac{g \times 1}{9+1} = 3 \cdot 2$;

therefore, the space described in 3 seconds $= 1 \cdot 6 \times 9 = 14 \cdot 4$ feet,
and the velocity acquired in 4 seconds $= 3 \cdot 2 \times 4 = 12 \cdot 8$.

The space described in the 5th second is $\frac{1}{2} f$ multiplied by the 5th odd number, or 9; therefore,

the space described in 5th second $= 1 \cdot 6 \times 9 = 14 \cdot 4$ feet.

Ex. 3. A weight of 10 lbs. draws a weight of 8 lbs. over a fixed pulley; required the tension in the string, and the time of raising the lighter weight to a height of 64 feet.

By Art. 174,

$$\text{the tension} = \frac{2 \times 10 \times 8}{10+8} = 8\tfrac{8}{9} \text{ lbs.}$$

Also, if f be the acceleration,

$$f = g \cdot \frac{10-8}{10+8} = \frac{g}{9}.$$

Substituting this value for f, and 64 for s, in the equation $s = \frac{1}{2} f t^2$, we have

$$64 = \frac{g}{18} t^2;$$

$$\therefore \qquad t^2 = 36 \text{ nearly,}$$

$$\text{or,} \qquad t = 6 \text{ seconds.}$$

176. *If a body be projected with a given velocity* V, *and be acted on in the same direction by a uniform force, to find the velocity acquired and the space described in a given time* t.

From the second law of motion, it follows that the velocity will equal the velocity of projection, together with the velocity generated by the force; and, therefore, if f be the acceleration produced by the force, and v the velocity required,

$$v = V + ft. \qquad \text{(i.)}$$

And similarly, the space described is equal to the space due to the velocity of projection, together with the space due to the action of the force. The space described by the uniform velocity V in t seconds is equal to Vt, and the space due to the action of the force in t seconds is equal to $\frac{1}{2}ft^2$. Therefore,

$$s = Vt + \tfrac{1}{2}ft^2. \qquad \text{(ii.)}$$

Eliminating t between equations i. and ii. we have

$$v^2 = V^2 + 2fs. \qquad \text{(iii.)}$$

177. If a body be projected with a given velocity V in a direction opposite to that in which the force acts, the expressions deduced, as in the preceding Article, will be

$$v = V - ft,$$
$$s = Vt - \tfrac{1}{2}ft^2.$$
$$v^2 = V^2 - 2fs.$$

178. *To find the height to which a body will rise when projected vertically with a given velocity* V.

Here the direction of V is opposite to that of the force of gravity, and hence, if v denote the velocity acquired when the body has described the space s, by the preceding Article,

$$v^2 = V^2 - 2gs.$$

But when the body has attained the highest point, its velocity at that instant will be zero; and, therefore, if s be the height of this point,

$$0 = V^2 - 2gs;$$
$$\therefore \quad 2gs = V^2.$$

By comparing this result with the expression $v^2 = 2gs$ in Art. 169, we see that the height to which the body will rise is the same as the distance through which it must fall to acquire the velocity of projection.

179. *To find the time during which a body will rise when projected vertically with a given velocity* **V.**

Since at the instant of attaining the greatest height $v = 0$, then (Art. 177),

$$V - gt = 0;$$

$$\therefore \quad gt = V,$$

$$\therefore \quad t = \frac{V}{g}.$$

From the general formula $v = gt$, it follows that $\frac{V}{g}$ is also the time during which a body must fall to acquire the velocity V, and hence from this and the preceding section we see that, on its return to the point whence it was projected, the body will have a velocity equal to the velocity of projection, and that the times of descent and ascent will be equal.

180. The following are examples in illustration of Articles 173–6.

Ex. 1. A body is projected vertically upwards with a velocity of 100 feet per second, to find how high it will rise.

By Art. 178, $2gs = V^2$; therefore, in this case,

$$s = \frac{10000}{64.4} = 155.28.$$

Ex. 2. A body is projected vertically upwards with a velocity of 161 feet per second, to find how long it will continue to ascend.

By Art. 179, $t = \frac{V}{g}$, therefore in this case,

$$t = \frac{161}{32.2} = 5 \text{ seconds.}$$

Ex. 3. A stone is projected vertically upwards, and returns to the same spot after an interval of 12 seconds; find the velocity of projection, and the height to which the body has risen.

The velocity of projection is equal to the velocity acquired by the body during the time of descent, or 6 seconds; and, therefore, if V be the velocity of projection, since $v = gt$,

$$V = 32{\cdot}2 \times 6 = 193{\cdot}2 \text{ feet per second.}$$

The height to which the body has risen is equal to the distance through which it falls from rest during six seconds; that is,

$$\begin{aligned}\text{the height required} &= \tfrac{1}{2}g \times 6^2,\\ &= 16{\cdot}1 \times 36 = 579{\cdot}6 \text{ feet.}\end{aligned}$$

Ex. 4. A ball is projected vertically with a velocity 160 feet per second, and two seconds afterwards another ball is projected in the same direction with a velocity 224; when and at what height will the balls meet?

Let the balls meet x seconds after the projection of the second ball, then $x + 2$ is the time the first ball has been moving.

The space described by the first ball in $x+2$ seconds is (Art. 174,)

$$160\,(x+2) - 16\,(x+2)^2.$$

Similarly, the space described by the second ball in x seconds is

$$224x - 16x^2.$$

But since the balls meet, these two quantities must be equal;

$$\begin{aligned}\therefore\quad 224x - 16x^2 &= 160\,(x+2) - 16\,(x+2)^2;\\ \therefore\quad 128x &= 256;\\ \therefore\quad x &= 2;\end{aligned}$$

or the balls meet 2 seconds after the projection of the second ball. To find the height, substitute 2 for x in either of the expressions for the space described by the balls; *e.g.*, $224x - 16x^2$, then

$$\text{height} = 224 \times 2 - 16 \times 4 = 384.$$

181. *To find the acceleration when a body moves down a smooth inclined plane, under the action of gravity.*

Let a weight W rest upon an inclined plane, whose length is l and whose height is h. W would be kept at rest on this plane by a force acting along the plane equal to $W\frac{h}{l}$ (Art. 95). Therefore

$W\frac{h}{l}$ is the pressure which causes the body to move down the plane. Hence, by Art. 154,

$$\text{acceleration} = g \, . \, \frac{W\frac{h}{l}}{W} = \frac{gh}{l}.$$

182. *Required the time of descent when a body falls down a smooth inclined plane.*

By the general formula, $s = \frac{1}{2}ft^2$. Let h = height of the plane, and l = its length. Then, in this case, $s = l$, and by the preceding Article, $f = \frac{gh}{l}$, and therefore

$$l = \frac{gh}{2l} \, . \, t^2;$$

$$\therefore \quad t^2 = \frac{2l^2}{gh},$$

$$\therefore \quad t = l\sqrt{\left(\frac{2}{gh}\right)}.$$

183. *To find the velocity acquired by a body falling down a smooth inclined plane.*

By the general formula, $v = \sqrt{(2fs)}$. As in the preceding Article, $s = l$, and $f = \frac{gh}{l}$. Therefore

$$v = \sqrt{(2gh)}$$

But $\sqrt{(2gh)}$ expresses the velocity which a body acquires in falling freely under the action of gravity through a distance h; therefore the velocity acquired by a body falling down an inclined plane is the same as that acquired in falling through the perpendicular height. Consequently, whatever be the length of the plane, the velocity acquired is the same, if the height remains the same.

184. The results of the three preceding Articles may be also deduced as follows:—

Let α be the inclination of the plane, then the angle between the direction of gravity and the plane is $90° - \alpha$; and therefore by Art. 166, the effective acceleration is

$$g \sin \alpha.$$

To find the time of descent, substitute this value for f, and for s, in the general formula $s = \frac{1}{2}ft^2$; then

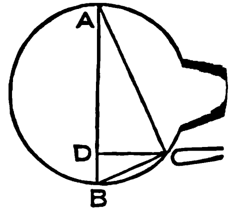
$$l = \tfrac{1}{2}g \sin \alpha \,.\, t^2;$$

therefore,
$$t = \sqrt{\left(\frac{2l}{g \sin \alpha}\right)},$$
$$= l\sqrt{\left(\frac{2}{gh}\right)}, \quad \text{since } \sin \alpha = \frac{h}{l}.$$

Also, if v be the velocity acquired in falling down the plane, substituting for s and f the values mentioned above in the formula $v = \sqrt{(2fs)}$,
$$v = \sqrt{(2g \sin \alpha \,.\, l)},$$
$$= \sqrt{(2gh)}.$$

185. *If a circle be placed with its plane vertical, the times of descent down all chords drawn through its highest or lowest points are equal.*

Let ABC be any circle whose plane is vertical. Let AB be the vertical diameter, and AC any chord drawn through A. Let t be the time in which a body falls down AC, then (Art. 182),



$$t = \sqrt{\left(\frac{2\,\mathrm{AC}^2}{g\,.\,\mathrm{AD}}\right)}.$$

The triangles ACD, ACB are similar, therefore

$$\mathrm{AD} : \mathrm{AC} :: \mathrm{AC} : \mathrm{AB}, \text{ or } \frac{\mathrm{AC}^2}{\mathrm{AD}} = \mathrm{AB};$$

$$\therefore \qquad t = \sqrt{\left(\frac{2\,\mathrm{AB}}{g}\right)}.$$

As this result is independent of the position of the point C, it follows that the times of descent down all chords drawn through A are equal.

And similarly, if BC be any chord drawn through B, the time of descent is

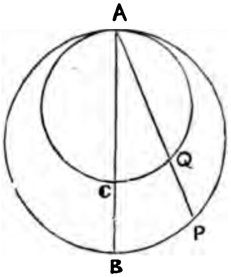
$$\sqrt{\left(\frac{2 \cdot BC^2}{g \cdot BD}\right)}.$$

But $\frac{BC^2}{BD} = AB$; therefore, as before, the time of descent is

$$\sqrt{\left(\frac{2 \cdot AB}{g}\right)}.$$

186. *If two circles, the planes of which are vertical, touch one another internally at their highest points, and any chord of the larger circle be drawn through the point of contact, the time of descent down that part of the chord which is exterior to the smaller circle is the same for all such lines.*

Let the two circles touch one another internally at A, and let A be their highest point. Let AB be a diameter of the larger circle, and AP any chord.



Let T be the time of descent down AP, and t the time of descent down AQ, then, by Art. 172, the time of descent down QP is equal to

$$\sqrt{(T^2 - t^2)}.$$

But, by Art. 185, T is the time of descent down AB, and t is the time of descent down AC; therefore the time of descent down CB is equal to

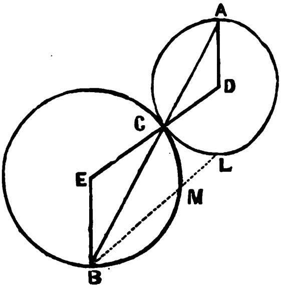
$$\sqrt{(T^2 - t^2)}.$$

Hence, the time of descent down QP is equal to the time of descent down CB.

187. In a similar manner it may be shown, that if two circles touch one another internally in their *lowest* points, and any chord of the larger circle be drawn through the point of contact, the time of descent down that portion of the chord which is exterior to the smaller circle is invariable.

188. By aid of the property of the circle demonstrated in Art. 185, many problems relating to planes of quickest and slowest descent may be readily solved. The following are examples.

PROB. 1. *To find the line of quickest descent from a circle to a point without it, the point and the circle being in the same vertical plane, and the point lower than the highest point of the circle.*



Let AC be the given circle, having its centre at D, and its highest point at A, and let B be the given point. Join the points A, B; the line BC is the line required.

Through B draw a line parallel to AD, and meeting DC produced in E. Then, since the triangle ACD is isosceles,

$$\angle ACD = \angle CAD;$$

and, since the lines BE, AD are parallel,

$$\angle EBC = \angle CAD;$$

therefore, $\angle EBC = \angle ACD = \angle ECB;$

and $EC = EB.$

Hence, a circle described with E as a centre and EB as a radius will touch the given circle in C; and, since EB is vertical, it will have its lowest point at B.

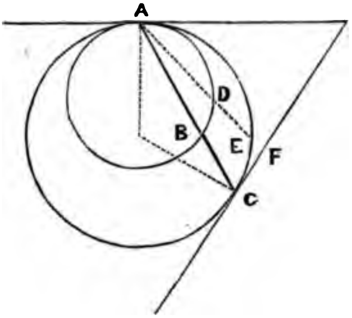
Consequently, the time of descent down CB is less than that down any other line drawn from B to the given circle. For let LB be some other line. The time of descent down CB is equal to the time of descent down MB, and is therefore less than the time of descent down LB.

PROB. 2. *To find the line of quickest descent from a given circle to a given line without it, the line and the circle being in the same vertical plane.*

Let ABD be the given circle, having its highest point at A, and CF the given line.

Through A draw AC, making equal angles with CF and the horizontal line through A; then BC shall be the line required.

Through A and C draw a circle ACE, having its highest point at A. It can easily be shown that this circle will touch the line CF in C.

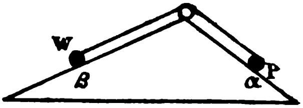


By the previous problem, BC is the line of quickest descent from the given circle to the point C. In like manner DF is the line of quickest descent from the given circle to any other point F in the line CF.

The time of descent down DE is less than the time of descent down DF; but, by Art. 186, the time of descent down BC is equal to that down DE. Therefore, the time of descent down BC is less than down DF. Consequently, BC is the line of quickest descent from the given circle ABD to the given line CF.

189. *A weight* P *placed on a smooth inclined plane is connected by a cord with another weight* W *placed on another smooth inclined plane, the two planes meet in their highest point; the cord passes over a pulley placed at the common summit: to find the acceleration and the tension in the string.*

Let α and β be the inclination of the planes, and f the acceleration, T the tension in the cord, and let P be supposed to be descending and W to be ascending.



The force causing motion in P is $P \sin \alpha - T$, thence, Art. 154,

$$\frac{f}{g} = \frac{P \sin \alpha - T}{P}.$$

The force causing motion in W is $T - W \sin \beta$, and hence, since the acceleration in both weights is the same,

$$\frac{f}{g} = \frac{T - W \sin \beta}{W}.$$

Solving these equations, we obtain

$$T=\frac{PW(\sin\alpha+\sin\beta)}{P+W},$$

and

$$f=g\frac{(P\sin\alpha-W\sin\beta)}{P+W}.$$

Cor. 1. If $\alpha=90^\circ$, we have the case of one weight hang freely and drawing another up an inclined plane. In this case

$$T=\frac{PW(1+\sin\beta)}{P+W},\ \text{and}\ f=g\frac{(P-W\sin\beta)}{P+W}.$$

Cor. 2. If $\beta=0$, we have the case of one weight moving do an inclined and drawing another along a horizontal plane. In t case

$$T=\frac{PW\sin\alpha}{P+W},\ \text{and}\ f=g\frac{P\sin\alpha}{P+W}.$$

Cor. 3. If $\alpha=90^\circ$ and $\beta=90^\circ$, we have the case of both bo hanging freely, and, as already seen in Art. 174,

$$T=\frac{2PW}{P+W},\ \text{and}\ f=g\frac{(P-W)}{P+W}.$$

190. In the preceding problem let the planes be rough, and μ and μ' be the coefficients of friction.

Then the force causing motion in P is (Art. 122) P(sin $\alpha-\mu$ cos $-$ T, thence, Art. 154,

$$\frac{f}{g}=\frac{P(\sin\alpha-\mu\cos\alpha)-T}{P}.$$

The force causing motion in W is $T-W(\sin\beta+\mu'\cos\beta)$, a hence

$$\frac{f}{g}=\frac{T-W(\sin\beta+\mu'\cos\beta)}{W};$$

whence

$$T=\frac{PW(\sin\alpha-\mu\cos\alpha+\sin\beta+\mu'\cos\beta)}{P+W},$$

and

$$f=\frac{g[P(\sin\alpha-\mu\cos\alpha)-W(\sin\beta+\mu'\cos\beta)]}{P+W}.$$

Cor. 1. If $W=0$, we have the case of a body moving by own weight down a rough inclined plane. In this case

$$f=g(\sin\alpha-\mu\cos\alpha).$$

COR. 2. If $\alpha = 90^\circ$, we have the case of a body hanging freely and drawing another up a rough inclined plane. In this case

$$T = \frac{PW(1 + \sin\beta + \mu' \cos\beta)}{P + W}.$$

$$f = \frac{g[P - W(\sin\beta + \mu' \cos\beta)]}{P + W}.$$

COR. 3. If $\alpha = 90^\circ$ and $\beta = 0$, we have the case of a body hanging freely and drawing another along a rough horizontal plane. In this case

$$T = \frac{PW(1 + \mu')}{P + W},$$

$$f = \frac{g(P - \mu' W)}{P + W}.$$

COR. 4. If $\beta = 90^\circ$, we have the case of a body moving down a rough inclined plane and drawing up another hanging freely. In this case

$$T = \frac{PW(\sin\alpha - \mu\cos\alpha + 1)}{P + W},$$

$$f = g\frac{[P(\sin\alpha - \mu\cos\alpha) - W]}{P + W}.$$

191. It will be seen from the preceding problem that the retardation arising from friction when a body moves along an inclined plane is $\mu g \cos\alpha$; that this is contrary to the acceleration due to gravity in the case of a body moving down the plane, but is with it when moving up the plane; and that hence, in the former case, the acceleration is $g(\sin\alpha - \mu\cos\alpha)$, but in the latter it is $g(\sin\alpha + \mu\cos\alpha)$.

These accelerations may also be written $\frac{g \,.\, \sin(\alpha - \epsilon)}{\cos\epsilon}$ and $\frac{g \,.\, \sin(\alpha + \epsilon)}{\cos\epsilon}$ where $\tan\epsilon = \mu$.

EXAMPLES.

1. If a body be projected vertically upwards with a velocity of 128·8 feet per second, when will it attain the height of 241·5?
Ans. 3 and 5 seconds.

2. A body projected vertically upwards is at the same height at the expiration of 4 and 5 seconds, what was its greatest elevation?
Ans. 326 feet.

3. A body projected vertically upwards is at the same height at the expiration of 2 and 3 seconds, required the velocity of projection. Ans. 80·5 feet.

4. A ball is projected vertically upwards with a velocity of 161 feet per second, when must another ball be projected in the same direction with a velocity 225·4, in order that the balls may meet at the height of 386·4 feet? Ans. The second ball must be projected 2 or 4 seconds *after* the first, or 6 or 8 seconds *before*.

5. Two balls are projected vertically at the same instant, one upwards with a velocity V_1, and the other down with a velocity V_2, when will they be equally distant from the point of projection?
Ans. In $\frac{V_1 - V_2}{g}$ seconds.

6. A ball is projected horizontally with a velocity V, required its velocity at the end of t seconds. Ans. $(\sqrt{V^2 + g^2t^2})$.

7. A body slides by its own weight down a perfectly smooth inclined plane, rising 5 in 48; what is the space described in 3 seconds? Ans. 15 feet.

8. A body slides by its own weight down a smooth inclined plane, whose length is 100 feet and height 9; what is the time of its descent? Ans. $8\frac{1}{3}$ seconds nearly.

9. A body is projected with a velocity of 16·1 feet per second, up a smooth inclined plane rising 1 in 5, required the time and length of its ascent.

The time of ascent is $2\frac{1}{2}$ seconds, and the length $20\frac{1}{8}$ feet.

10. A body slides by its own weight down a rough inclined plane rising 7 in 25, what is the space described in 5 seconds, the coefficient of friction being $\frac{1}{4}$. Ans. 16·1 feet.

11. A body is projected along a rough horizontal plane with a velocity of 100 feet per second, how far will it go before coming to rest, the coefficient of friction being $\frac{2}{7}$? Ans. 543·5 ft. nearly.

12. A body is projected up a smooth inclined plane, whose length is 30 feet and height 8 feet; what must be the velocity of projection that it may just reach the top of the plane?

Ans. 22·7 nearly.

13. With what velocity must a body be projected up a rough inclined plane, whose length is 82 and height 18, that it may just reach the top, the coefficient of friction being $\frac{7}{40}$.

Ans. 45·4 feet nearly.

14. In the preceding what velocity will a body acquire in falling down the plane? Ans. 16 feet.

15. If V_1 be the velocity required to be given to a body that it may rise to the top of a rough inclined plane, and V_2 the velocity acquired by falling down the same plane, then, if a be the inclination of the plane, the coefficient of friction is

$$\frac{(V_1^2 - V_2^2)}{V_1^2 + V_2^2} \tan a.$$

16. From a given point draw a straight line down which a body will descend to a fixed line in the shortest time possible.

17. Two circles, the planes of which are vertical, are so placed that the lowest point of one is in contact with the highest point of

the other; show that the time of descent from any point in the former to any point in the latter, along a straight line passing through the point of contact, is invariable.

18. A body after falling down a smooth inclined plane whose height is h and length l, moves along a smooth horizontal plane, with what velocity does it move? Ans. $\frac{\sqrt{\{2gh(l^2-h^2)\}}}{l}$.

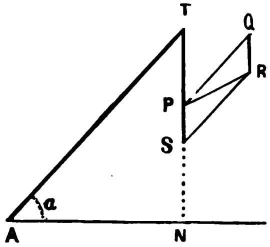
19. A body after falling down a rough inclined plane, whose length is l and inclination α, moves along an equally rough horizontal plane, how far will it move before it comes to rest, supposing ϵ to equal the angle of repose? Ans. $\frac{l \sin(\alpha-\epsilon)\cos^2\alpha}{\sin\epsilon}$.

CHAPTER XI.

ON PROJECTILES.

DEF. The *angle of elevation* of a projectile is the inclina- the direction of projection to the horizontal plane through t of projection.

A body is projected with a given velocity V, *at a given angle ion* α, *to determine the place of the body and the magnitude ction of its velocity at the end of* t''.



. be the point, and AT the direction of projection. Make , and through T draw TP vertically downwards, and equal or the space due to the f gravity in t''. Then, by nd law of motion, P will be e of the body at the end

gh P draw PQ parallel to equal to V and PS, vertically rds, and equal to gt, or the generated by gravity in t''. e the parallelogram QS, then, by Art. 161, the diagonal esents the required velocity in magnitude and direction. resent this velocity, and θ the angle between its direction downward vertical line, then

$$v^2 = PQ^2 + QR^2 - 2PQ \,.\, QR \cos PQR,$$

$$= V^2 + g^2t^2 - 2Vgt \sin \alpha.$$

$$\sin \theta = \frac{PQ \sin PQR}{PR},$$

$$= \frac{V \cos \alpha}{v}.$$

194. *To determine the least velocity of a projectile, and the time of attaining it.*

Let V be the velocity of projection, and α the angle of elevation, then, by the preceding, the square of the velocity at any time t is

$$V^2 + g^2t^2 - 2Vgt \sin \alpha.$$

By substituting $V^2(\cos^2\alpha + \sin^2\alpha)$ for V^2, this expression may be written in the form

$$V^2 \cos^2 \alpha + (gt - V \sin \alpha)^2.$$

This expression has its least value when its second term is equal to zero, or $gt = V \sin \alpha$; hence,

$$\text{least velocity} = V \cos \alpha,$$

$$\text{time of reaching it} = \frac{V \sin \alpha}{g}.$$

COR. The value of the least velocity just found is equal to the horizontal component of the velocity of projection; whence it may be inferred that the body is moving in a horizontal direction at the instant of attaining its least velocity. This may also be deduced from the expression already found for determining the direction of the velocity. For, substituting V cos α for v in the expression

$$\sin \theta = \frac{V \cos \alpha}{v}, \text{ we have}$$

$$\sin \theta = \frac{V \cos \alpha}{V \cos \alpha} = 1,$$

$$\therefore \quad \theta = 90°;$$

or the direction of the velocity is inclined to the vertical line at an angle of 90°; that is, is horizontal.

195. *To determine the elevation of a projectile at any given instant of time.*

Let V be the velocity of projection, and α the angle of elevation. Then, in fig. Art. 193, PN represents the elevation at the end of t''.

$$\text{Elevation} = TN - TP,$$

$$= Vt \sin \alpha - \tfrac{1}{2}gt^2.$$

196. *To determine the greatest elevation of a projectile, and the time of reaching it.*

By the preceding,

$$\text{elevation} = \frac{2Vgt\sin\alpha - g^2t^2}{2g}.$$

$$= \frac{V^2\sin^2\alpha - (V\sin\alpha - gt)^2}{2g}.$$

This expression has its greatest value when the second term is equal to zero or $gt = V\sin\alpha$; hence

$$\text{the greatest elevation} = \frac{V^2\sin^2\alpha}{2g},$$

$$\text{the time of reaching it} = \frac{V\sin\alpha}{g}.$$

COR. By a comparison of this result with Art. 194, it is seen that at the instant of reaching the greatest elevation, the projectile is moving with its least velocity.

197. DEFS. The *range* of a projectile is the distance between the point of projection and the point where the body strikes any plane passing through the point of projection; and the *time of flight* is the time the body takes in describing its path between these two points.

198. *To find the time of flight and the range of a projectile on a horizontal plane.*

Let V be the velocity of projection, and α the angle of elevation. Then, by Art. 195, the elevation of the body at the time t is

$$Vt\sin\alpha - \tfrac{1}{2}gt^2.$$

This expression is equal to zero, or the body is in the horizontal plane through the point of projection when either $t = 0$, or $t = \frac{2V\sin\alpha}{g}$. The former of these values denotes the instant of projection; the latter, the instant of returning to the horizontal plane. Hence,

$$\text{time of flight} = \frac{2V\sin\alpha}{g}.$$

From fig. Art. 193, it will be seen that AN, or the horizontal distance described in the time t, is equal to $Vt \cos a$; therefore,

$$\begin{aligned} \text{horizontal range} &= V \cos a \times \text{time of flight}, \\ &= \frac{2V^2 \sin a \cos a}{g}, \\ &= \frac{V^2 \sin 2a}{g}. \end{aligned}$$

COR. 1. By comparing the results of the present section with those of Art. 196, it will be seen that the time of flight is double the time of reaching the greatest elevation, and consequently that the times of ascent and descent are equal. Also, that the ratio of the horizontal range to the greatest elevation is constant for the same angle of elevation, being equal to

$$4 \cot a.$$

COR. 2. Since $\sin 2a$ has its greatest value when $a = 45°$, it follows that, with a given velocity of projection, the horizontal range is the greatest when the angle of elevation is 45°, and hence, V being the velocity of projection,

$$\text{greatest horizontal range} = \frac{V^2}{g}.$$

199. Some of the preceding results may be simplified by the following device: Let h represent the distance through which a body must fall under the action of gravity to acquire the velocity of projection, then, Art. 169, $V^2 = 2gh$, and hence

$$\begin{aligned} \text{greatest elevation} &= h \sin^2 a, \\ \text{horizontal range} &= 2h \sin 2a, \\ \text{greatest horizontal range} &= 2h. \end{aligned}$$

200. *Given the time of flight of a projectile, on a horizontal plane, to find the greatest elevation.*

Let T be the time of flight, then

$$\frac{2V \sin a}{g} = T,$$

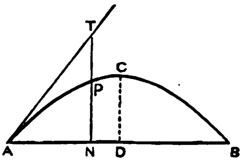
$$V \sin a = \tfrac{1}{2} gT;$$

$$\begin{aligned} \text{greatest elevation} &= \frac{V^2 \sin^2 a}{2g}, \\ &= \frac{gT^2}{8}. \end{aligned}$$

e all projectiles, having equal times of flight on a horizontal , attain to equal elevations; and, conversely, if the greatest tions are equal, the times of flight are equal also.

1. *To determine the path described under the action of gravity body projected with a given velocity in a given direction.*

t the body be projected from A in the direction AT, with the ity V. Let AT be the distance due e velocity of projection in any time t, ce AT $=$ Vt. Draw the vertical line gh T, and let TP $= \frac{1}{2}gt^2 =$ the space to the action of gravity in the time t. a, by the second law of motion, P will he place of the body at the end of the time t.



et y and x be the co-ordinates of P, referred to AT and the ical line through A as axes. Then

$$y = \text{AT} = \text{V}t,$$

$$x = \text{TP} = \tfrac{1}{2}gt^2;$$

. eliminating t $$y^2 = \frac{2\text{V}^2}{g}x,$$

he path of the projectile is a parabola.

02. *To find the equation to the path of a projectile when the of co-ordinates are the horizontal and vertical lines drawn ugh the point of projection.*

et AN, NP (fig. Art. 201) be the co-ordinates of P, and let $= y$, NP $= x$, and the angle TAN $= \alpha$,

$$y = \text{AT} \cos \alpha = \text{V}t \cos \alpha,$$

$$x = \text{TN} - \text{TP} = \text{V}t \sin \alpha - \tfrac{1}{2}gt^2;$$

∴ eliminating t,

$$x = \frac{\sin \alpha \,.\, y}{\cos \alpha} - \frac{gy^2}{2\text{V}^2 \cos^2 \alpha},$$

or, $$y^2 - \frac{\text{V}^2 \sin 2\alpha}{g} \,.\, y = \frac{2\text{V}^2 \cos^2 \alpha}{g} \,.\, x;$$

nce it appears that the path of the projectile is the parabola se *latus rectum* is $\frac{2\text{V}^2 \cos^2 \alpha}{g}$.

203. If, as before, h be the distance through which a body must fall vertically under the action of gravity to obtain the velocity V, then since $V^2 = 2gh$, the equation to the path of the projectile becomes $y^2 - 2h \sin 2\alpha . y = -4h \cos^2 \alpha . x.$

Again, if h' be the distance through which a body must fall to acquire a velocity equal to the horizontal component of the velocity of projection, or $V \cos \alpha$, so that $V^2 \cos^2 \alpha = 2gh'$, the preceding equation becomes $y^2 - 4h' \tan \alpha . y = -4h' . x.$

Hence, the *latus rectum* of the parabola described by a projectile is four times the space through which a body must fall to acquire the horizontal component of the velocity of projection.

204. The values already obtained (Art. 199) for the greatest elevation and the horizontal range, may also be deduced immediately from the equation to the path of the projection. By Art. 203,

$$y^2 - 2h \sin 2\alpha . y = -4h \cos^2 \alpha . x;$$

$$\therefore \quad (y - h \sin 2\alpha)^2 = -4h \cos^2 \alpha . x + h^2 \sin^2 2\alpha,$$

$$= -4h \cos^2 \alpha \, (x - h \sin^2 \alpha),$$

which is the equation to a parabola, having its vertex at a point whose co-ordinates are $h \sin^2 \alpha$ and $h \sin 2\alpha$. Therefore (fig. Art. 201),

$$CD = h \sin^2 \alpha,$$

$$AD = h \sin 2\alpha;$$

but CD is the greatest elevation, and AD is half the horizontal range.

205. *The distance of the point of projection from the directrix of the parabolic path is equal to the space due to the velocity of projection.*

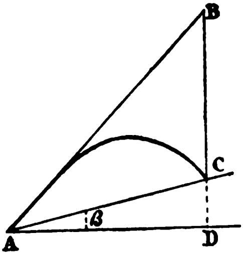
The distance of the point of projection from the directrix is equal to the distance between the directrix and the horizontal plane through the point of projection. The *latus rectum* of the parabolic path is $4h \cos^2 \alpha$, and consequently the distance of the directrix from the vertex is $h \cos^2 \alpha$. But the distance of the vertex from the horizontal plane is $h \sin^2 \alpha$; therefore

$$\text{height of directrix} = h \cos^2 \alpha + h \sin^2 \alpha,$$

$$= h.$$

Any point in the path of a projectile may be regarded as the point of projection, if the body be supposed to be projected from it with the velocity and in the direction which it has at that point. Hence it follows from the preceding, that the velocity of a projectile at any point in its path is equal to that acquired by a body falling to the point from the directrix.

206. *To find the range and time of flight of a projectile on an inclined plane.*



Let the body be projected from A in the direction AB, and with the velocity V. Let β be the inclination of the plane, and α the inclination of AB to the horizontal line. Let the body strike the plane at C. Through C draw the vertical line BC, meeting AB in B. Then if T be the time of flight, $AB = VT$, and $BC = \frac{1}{2}g\,T^2$.

But
$$\frac{BC}{AB} = \frac{\sin BAC}{\sin BCA},$$

$$\therefore \quad \frac{\frac{1}{2}gT^2}{VT} = \frac{\sin(\alpha - \beta)}{\cos\beta},$$

or
$$T = \frac{2V\sin(\alpha-\beta)}{\cos\beta}. \quad \checkmark$$

Again, since $AC\cos B = AD = AB\cos\alpha$,

$$AC = \frac{VT\cos\alpha}{\cos\beta},$$

$$= \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{\cos^2 B}.$$

207. *To find the direction in which a body must be projected with a given velocity in order to strike a given fixed mark.*

Let C (fig. Art. 206) be the mark, and let $AD = b$, and $CD = c$. Let V be the velocity of projection, and α the angle its direction makes with the horizontal line. Then, if the mark be struck at the end of the time t,

$$V \cos a \, . \, t = b,$$

and

$$V \sin a \, . \, t - \tfrac{1}{2} g t^2 = c.$$

Therefore, eliminating t,

$$b \tan a - \frac{1}{2} \frac{g b^2}{V^2 \cos^2 a} \doteq c.$$

Whence,

$$b \tan a - \frac{g}{2} \frac{b^2}{V^2} (1 + \tan^2 a) = c,$$

$$g b \tan^2 a - 2 V^2 . \tan a + \frac{2 V^2 c + g b^2}{b} = 0 ;$$

$$\therefore \qquad \tan a = \frac{V^2 \pm \sqrt{(V^4 - 2 V^2 c g - g^2 b^2)}}{b}.$$

Or, if $2 g h = V^2$,

$$\tan a = \frac{2 h \pm \sqrt{(4 h^2 - 4 h c - b^2)}}{b}.$$

EXAMPLES.

1. The horizontal range of a projectile is 1000 feet, and the time of flight 10 seconds; required the velocity of projection.

Ans. 189.6 ft. per second.

2. The greatest elevation attained by a projectile is equal to its horizontal range; required the angle of elevation. Ans. $\tan^{-1} 4$.

3. In what direction must a body be projected with a velocity V, that its range on a plane whose inclination is β may be the greatest possible? Ans. $45° + \frac{1}{2}\beta$.

4. Show that the greatest possible range, on a plane whose inclination is β, of a body projected with a velocity due to the height h is

$$\frac{2h}{1 + \sin \beta}.$$

5. The horizontal range of a projectile is equal to the space due to the velocity of projection; what is the angle of elevation?

If a body be projected from a height h' above a horizontal e, with a velocity due to the height h, and at an angle of tion equal to α, what is its range along the plane?

Ans. $h \sin 2\alpha + 2 \cos \alpha\sqrt{(h^2 \sin^2 \alpha + hh')}$.

If any number of bodies be projected with the same velocity different directions from the same point, show that their locus e end of the time t is the sphere whose radius is vt, and whose re is at a distance $\frac{1}{2}gt^2$ vertically below the point of projection.

. Two bodies projected from the same point, at angles of eleon α_1, α_2, respectively, and with velocities due to the heights h_2, strike an inclined plane at the same point, find the inclina of the plane.

Ans. $\tan^{-1}\left(\frac{1}{2} \cdot \frac{h_1 \sin 2\alpha_1 - h_2 \sin 2\alpha_2}{h_1 \cos^2 \alpha_1 - h_2 \cos^2 \alpha}\right)$.

. Two bodies projected from the same point, at angles of eleion α_1, α_2, respectively, and with velocities due to the heights h_2, strike the same point in a vertical wall, find the distance of wall from the point of projection.

Ans. $4h_1h_2 \cdot \frac{\tan \alpha_1 - \tan \alpha_2}{h_2 \sec^2 \alpha_1 - h_1 \sec^2 \alpha_2}$.

10. A ball is projected with a velocity of 966 feet; what must the angle of elevation that its horizontal range may be 9660 ards? Ans. 45°.

11. The greatest horizontal range of a projectile is 805 feet; hat was its velocity of projection? Ans. 161 ft. per sec.

12. A ball is projected with a velocity acquired by falling over 100 feet; what must be the lesser angle of elevation that it may hit a mark 64 feet high and distant (horizontally) 320 feet?

Ans. 45°.

13. Show that the same ball, at the same elevation, would hit a mark at the same height, distant 80 feet.

14. At what horizontal distance will a ball projected at an elevation of 30° with the velocity acquired by falling through 144 feet, hit a mark 30 feet high? Ans. $36(2\sqrt{3} \pm \sqrt{2})$ ft.

15. Given the horizontal range and the time of flight, to determine the angle of elevation.

Let R be the horizontal range and T the time of flight, then, if a be the angle of elevation,

$$\tan a = \frac{gT^2}{2R}.$$

16. If any number of bodies be projected from the same point in the same direction, but with different velocities, show that the locus of their points of greatest elevation is a straight line through the point of projection inclined to the horizontal line at an angle whose tangent is $\frac{1}{2}$ tan a (a being the angle of elevation of the projectiles).

17. Two bodies are projected at the same instant from different points in the same horizontal plane; the sines of the angles of elevation are inversely proportional to the velocities of projection; show that their elevations are equal at every instant of their flight.

18. If two bodies, projected at the same instant from different points in the same horizontal plane, have at any instant equal elevations, show that their elevations are equal at every instant.

19. If any number of bodies be projected from the same point with the same velocity in different directions in the same vertical plane, show that the locus of their points of greatest elevation is an ellipse whose semi-major axis is h and semi-minor axis $\frac{1}{2}h$ (h being the height through which a body must fall to acquire the velocity of projection.

20. In the preceding, show that the locus of the foci of the parabolic paths is a circle whose centre is at the point of projection, and whose radius is equal to h.

CHAPTER XII.

ON IMPACT.

8. When any two bodies come into collision, the internal sition of their particles is disturbed to a greater or less ex-

It is found by experience, that when any such disturbance s place, all matter possesses the power of returning in some sure to its original state. To this property the name *elasticity* iven. Its effect when one body impinges upon another is ause them to separate, and the magnitude of the force is sured by the ratio which the velocity of the recoil bears to velocity of the approach. The value of this ratio is termed measure or modulus of the elasticity. When this ratio is y—that is, when the velocity of the recoil equals the velocity he approach—the elasticity is said to be *perfect;* in all other s it is called *imperfect.*

et V be the velocity of any body impinging upon another ing in the same direction with the velocity V′, and let v and v' heir velocities immediately after impact, then $V - V'$ is the city of approach, and $v' - v$ is the velocity of rebound. Hence be the modulus of elasticity

$$v' - v = e(V - V'),$$

or

$$v - v' = -e(V - V').$$

9. It has been seen (Art. 153) that the momenta generated in l times by equal forces are equal. When the forces producing on are the mutual action of two bodies in contact, they are l in magnitude and opposite in direction. Hence in such s the momenta generated are equal and opposite. Consequently, he direct impact of two bodies the algebraic sum of the mo-

menta is the same after impact as before. Hence if M and M′ are the masses of two bodies, V and V′ their velocities immediately before impact, and v and v' their velocities immediately after impact

$$vM + v'M' = VM + V'M'.$$

210. *Two smooth spherical and inelastic bodies impinge directly upon each other, to determine their velocity immediately after impact.*

In this case $e = o$, and therefore (Art. 208) $v' = v$, and hence, by the preceding Article,

$$v = \frac{MV + M'V'}{M + M'}.$$

If the bodies be moving in opposite directions, V′ will be negative, or

$$v = \frac{MV - M'V'}{M + M'}.$$

and hence if $MV = M'V'$, $v = o$ or the bodies will be at rest after impact.

If M impinge upon M′ at rest, $V' = o$, and

$$v = \frac{MV}{M + M'}.$$

211. *Two elastic spherical bodies impinge directly upon each other, to find their velocities immediately after impact.*

Let the two bodies move in the same direction with velocities V, V′, and let M, M′ be their masses. Let e be the modulus of elasticity, and v, v' the velocities of the bodies immediately after impact. Then, by Articles 208, 209,

$$v - v' = -e(V - V'),$$

$$vM + v'M' = VM + V'M';$$

whence
$$v = \frac{MV + M'V' - eM'(V' - V')}{M + M'},$$

and
$$v' = \frac{MV + M'V' + eM(V - V')}{M + M'},$$

If the two bodies move in contrary directions, let V′ be negative; then,

$$v = \frac{MV - M'V' - eM'(V + V')}{M + M'},$$

and
$$v' = \frac{MV - M'V' + eM(V + V')}{M + M'}.$$

COR. i. If $V' = 0$, M impinges upon M′ at rest; hence in this case

$$v = \frac{V(M - eM')}{M + M'},$$

$$v' = \frac{MV(1 + e)}{M + M'}.$$

And if $M = eM'$, then $v = 0$, or the impinging body is at rest immediately after impact, and $v' = eV$, or M′ moves with the velocity eV.

COR. ii. If $M' = M$, and $e = 1$, then

$$v = \frac{V \pm V' - (V \mp V')}{2} = \pm V',$$

$$v' = \frac{V \pm V' + (V \mp V')}{2} = V,$$

and hence, if two perfectly elastic bodies of equal masses, moving in the same or contrary directions with any velocities, impinge upon each other, they mutually exchange velocities.

212. *Two smooth spherical and inelastic bodies impinge obliquely upon each other, to determine their velocities and the directions of their motions after impact.*

Let M and M′ be the two bodies, moving with velocities V, V′ in directions making with the common tangent at the point of contact the angles α, α'. The mutual pressures during impact act along the common normal. The velocities V, V′ may be resolved into velocities $V \cos \alpha$, $V' \cos \alpha'$ acting along the common tangent, and $V \sin \alpha$, $V' \sin \alpha'$ acting along the common normal. The former, since the bodies are smooth, will not be affected by the impact; the latter will each become, according to Art. 210,

$$\frac{MV \sin \alpha + M'V' \sin \alpha'}{M + M'}.$$

Hence, if v and v' be the velocities of the bodies after impact, acting in directions making with the common tangent th angles θ, θ',

$$v \cos\theta = V \cos\alpha,$$

$$v \sin\theta = \frac{MV \sin\alpha + M'V' \sin\alpha'}{M+M'},$$

Whence
$$v^2 = V^2 \cos^2\alpha + \left(\frac{MV \sin\alpha + M'V' \sin\alpha'}{M+M'}\right)^2,$$

and
$$\tan\theta = \frac{MV \sin\alpha + M'V' \sin\alpha'}{V \cos\alpha\,(M+M')}.$$

Similarly,
$$v'^2 = V'^2 \cos^2\alpha' + \left(\frac{MV \sin\alpha + M'V' \sin\alpha'}{M+M'}\right)^2,$$

and
$$\tan\theta' = \frac{MV \sin\alpha + M'V' \sin\alpha'}{V' \cos\alpha'\,(M+M')}.$$

213. Hence, if $V \cos\alpha = V' \cos\alpha'$, $\theta = \theta'$ and $v = v'$, or th bodies after impact will both move in the same direction and wi a common velocity.

214. Again, if the directions of V and V' fall upon differe sides of the common tangent, α' will be negative, and therefore

$$\tan\theta = \frac{MV \sin\alpha - M'V' \sin\alpha'}{V \cos\alpha\,(M+M')},$$

and
$$\tan\theta' = \frac{MV \sin\alpha - M'V' \sin\alpha'}{V' \cos\alpha'\,(M+M')},$$

Consequently, if $MV \sin\alpha = M'V' \sin\alpha'$, $\theta = 0$, and $\theta' = 0$; or th two bodies after impact will both move in the direction of th common tangent, and their velocities will be $V \cos\alpha$ and $V' \cos$ respectively.

215. Hence, also, if M impinge obliquely upon M' at rest making $V' = 0$,

$$v = V\left\{\cos^2\alpha + \frac{M^2 \sin^2\alpha}{(M+M')^2}\right\}^{\frac{1}{2}},$$

$$\tan\theta = \frac{M \tan\alpha}{M+M'},$$

$$v' = \frac{MV \sin\alpha}{M+M'},$$

$$\tan\theta' = \infty; \qquad \text{or } \theta' = 90^\circ.$$

216. In the last, let $M' = \infty$, and we have the case of a body pinging upon an immoveable surface. Dividing numerator and nominator of the values of v and $\tan\theta$ by M',

$$v = \frac{V}{\frac{M}{M'} + 1} \cdot \sqrt{\left(\frac{M^2}{M'^2} + 2\frac{M}{M'}\cos\alpha + \cos^2\alpha\right)},$$

d

$$\tan\theta = \frac{\frac{M}{M'}\tan\alpha}{\frac{M}{M'} + 1}.$$

.t if $M' = \infty, \frac{M}{M'} = 0$; therefore,

$$v = V\cos\alpha,$$

$$\tan\theta = 0,$$

the body moves along the common tangent with a velocity cos α; and hence, if the fixed surface be a plane, a body imging obliquely upon it will, after impact, move along the plane.

17. *Two elastic spherical bodies impinge obliquely upon each r, to find their velocities and the directions of their motions ediately after impact.*

et M, M' be the two bodies, moving with velocities V, V', in ctions making with the common tangent, at the point of con-, the angles α, α'. Let v, v' be the velocities after impact, and $'$ the angles which their directions make with the common ent.

s in Art. 212, V and V' may be resolved into velocities $V\cos\alpha$, os α', acting along the tangent; and $V\sin\alpha$, $V'\sin\alpha'$, acting g the normal. The former are unaltered by the impact; the er will become respectively (Art. 211),

$$\frac{MV\sin\alpha + M'V'\sin\alpha' - eM'(V\sin\alpha - V'\sin\alpha')}{M + M'},$$

l

$$\frac{MV\sin\alpha + M'V'\sin\alpha' + eM(V\sin\alpha - V'\sin\alpha')}{M + M'}.$$

Hence,

$$v \cos \theta = \mathrm{V} \cos \alpha,$$

$$v \sin \theta = \frac{\mathrm{MV} \sin \alpha + \mathrm{M'V'} \sin \alpha' - e\mathrm{M'} (\mathrm{V} \sin \alpha - \mathrm{V'} \sin \alpha')}{\mathrm{M} + \mathrm{M'}},$$

$$\therefore v^2 = \mathrm{V}^2 \cos^2 \alpha + \left\{ \frac{\mathrm{MV} \sin \alpha + \mathrm{M'V'} \sin \alpha' - e\mathrm{M'} (\mathrm{V} \sin \alpha - \mathrm{V'} \sin \alpha')}{\mathrm{M} + \mathrm{M'}} \right\}^2$$

$$\text{and } \tan \theta = \frac{\mathrm{MV} \sin \alpha + \mathrm{M'V'} \sin \alpha' - e\mathrm{M'} (\mathrm{V} \sin \alpha - \mathrm{V'} \sin \alpha')}{\mathrm{V} \cos \alpha \, (\mathrm{M} + \mathrm{M'})}.$$

Similarly,

$$v'^2 = \mathrm{V'}^2 \cos^2 \alpha' + \left\{ \frac{\mathrm{MV} \sin \alpha + \mathrm{M'V'} \sin \alpha' + e\mathrm{M} (\mathrm{V} \sin \alpha - \mathrm{V'} \sin \alpha')}{\mathrm{M} + \mathrm{M'}} \right\}^2,$$

$$\text{and } \tan \theta' = \frac{\mathrm{MV} \sin \alpha + \mathrm{M'V'} \sin \alpha' + e\mathrm{M} (\mathrm{V} \sin \alpha - \mathrm{V'} \sin \alpha')}{\mathrm{V'} \cos \alpha' \, (\mathrm{M} + \mathrm{M'})}.$$

Cor. If $\mathrm{V'} = 0$, M impinges obliquely upon M′ at rest; hence in this case

$$v^2 = \mathrm{V}^2 \cos^2 \alpha + \left\{ \frac{\mathrm{V} \sin \alpha \, (\mathrm{M} - e\mathrm{M'})}{\mathrm{M} + \mathrm{M'}} \right\}^2,$$

$$\tan \theta = \tan \alpha \, . \, \frac{\mathrm{M} - e\mathrm{M'}}{\mathrm{M} + \mathrm{M'}},$$

$$v' = \frac{\mathrm{MV} \sin \alpha \, (1 + e)}{\mathrm{M} + \mathrm{M'}},$$

$$\tan \theta' = \infty \,; \text{ or } \theta' = 90^\circ.$$

218. *An imperfectly elastic spherical body impinges obliquely upon an immoveable surface, to determine the velocity and the direction of the motion immediately after impact.*

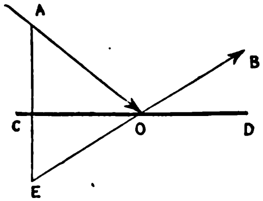
In the preceding corollary, let M′ be infinitely large in comparison with M, then $\frac{\mathrm{M}}{\mathrm{M'}}$ is indefinitely small.

Hence, $$v^2 = \mathrm{V}^2 (\cos^2 \alpha + e^2 \sin^2 \alpha),$$

$$\tan \theta = -e \tan \alpha.$$

Let AO be the direction of the motion of the impinging body, O the point at which it meets the surface, and CD the common tangent. From A, any point in AO, draw ACE perpendicular to CD, and take CE = eAC;

$$\begin{aligned}\tan \mathrm{BOD} &= \tan \mathrm{COE},\\ &= \frac{\mathrm{CE}}{\mathrm{CO}},\\ &= -\frac{e\mathrm{AC}}{\mathrm{CO}}\\ &= -e \tan \mathrm{AOC};\end{aligned}$$



therefore OB is the direction of the motion after impact.

219. *To determine the motion of the centre of gravity of two odies moving in the same straight line with uniform velocities.*

Let M, M′ be the masses of the two bodies, and V, V′ their elocities. Let a, a' be the distances of their centres of gravity rom any point in the line of their motion, and h the distance f their common centre of gravity from the same point; then Art. 64),

$$h = \frac{\mathrm{M}a + \mathrm{M}'a'}{\mathrm{M} + \mathrm{M}'}.$$

Let u be the velocity of the common centre of gravity, then at e expiration of any time t, a, a', and h will become $a + \mathrm{V}t$, $' + \mathrm{V}'t$, $h + ut$ respectively. Therefore, as before,

$$h + ut = \frac{\mathrm{M}(a + \mathrm{V}t) + \mathrm{M}'(a' + \mathrm{V}'t)}{\mathrm{M} + \mathrm{M}'}.$$

ubtracting the former from this we obtain

$$u = \frac{\mathrm{MV} + \mathrm{M}'\mathrm{V}'}{\mathrm{M} + \mathrm{M}'}.$$

220. *If two spherical bodies impinge directly, the velocity of heir centre of gravity immediately after impact is the same as mmediately before impact.*

Let M, M′ be the bodies, and V, V′ their velocities of impact. or the instant immediately preceding these velocities may be considered uniform; and therefore, by the preceding, if u denote the velocity of the centre of gravity,

$$u = \frac{MV + M'V'}{M + M'}.$$

If the bodies be inelastic, it has been already shown that their common velocity, and therefore the velocity of their centre of gravity, immediately after impact, is

$$\frac{MV + M'V'}{M + M'}.$$

The proposition is therefore true of inelastic bodies. If the bodies be elastic, and let v, v' be their velocities immediately after impact, these for the instant may be regarded as uniform; and therefore if u' denote the velocity of the centre of gravity,

$$u' = \frac{Mv + M'v'}{M + M'},$$

Substituting the values of v and v', from Art. 211,

$$\begin{aligned} u' &= \frac{M\{MV + M'V' - eM'(V - V')\} + M'\{MV + M'V' + eM(V - V')\}}{(M + M')^2}, \\ &= \frac{(MV + M'V')(M + M')}{(M + M')^2}, \\ &= \frac{MV + M'V'}{M + M'}, \\ &= u. \end{aligned}$$

221. The product of the mass of any body and the square of its velocity is called its *vis viva.*

222. *In the direct impact of imperfectly elastic bodies the sum of the* vires vivæ *after impact is less than before impact.*

By Art. 211,

$$v = \frac{MV + M'V' - eM'(V - V')}{M + M'},$$

$$v' = \frac{MV + M'V' + eM(V - V')}{M + M'};$$

$$\therefore \quad Mv^2 + M'v'^2 = \frac{(MV + M'V')^2 + e^2MM'(V - V')^2}{M + M'}.$$

In the numerator, add and subtract $MM'(V^2+V'^2)$. Then

$$Mv^2+M'v'^2=\frac{(MV^2+M'V'^2)(M+M')-(1-e^2)\,MM'(V-V')^2}{M+M'},$$

$$=MV^2+M'V'^2-\frac{(1-e^2)\,MM'(V-V')^2}{M+M'}.$$

COR. If the bodies be perfectly elastic, $e=1$, then

$$Mv^2+M'v'^2=MV^2+M'V'^2,$$

or the sum of the *vires vivæ* is the same after impact as before.

EXAMPLES.

1. A ball of 5 lbs. moving with a velocity 6 impinges directly upon a ball of 10 lbs. moving in the same direction with a velocity 4; the velocity of the latter after impact is 5, what is the modulus of elasticity? Ans. $\frac{1}{2}$.

2. A ball of 5 lbs. moving with a velocity 24 impinges directly upon a ball of 15 lbs. moving in the opposite direction with the velocity 8; what are the velocities of the balls after impact, the modulus of elasticity being $\frac{3}{4}$? Ans. 18 and 6.

3. A ball of 6 lbs. moving with a velocity 8 impinges directly upon a ball of $11\frac{1}{2}$ lbs. moving in the opposite direction with the velocity 6; show that if the modulus of elasticity be $\frac{1}{4}$, the latter ball is at rest after impact.

4. Three balls of equal weight are placed in a straight line upon a smooth horizontal plane; the first, moving with a velocity V, impinges on the second, and the second in turn on the third; with what velocity will the third ball move? Ans. $\frac{1}{4}(1+e)^2\,V$.

5. In the preceding, show that the first ball will impinge a second time upon the second ball. What will be the velocity of the first ball after the second impact? Ans. $\frac{1}{8}(1-e)(3+e^2)\,V$.

6. Two balls whose weights are as 4 to 1, moving in the same direction with velocities which are as 7 to 2, impinge directly; the faster ball is at rest after impact, show that this is also the lighter. What also is the modulus of elasticity? Ans. $\frac{3}{4}$.

7. Two balls whose weights are as 5 to 2, moving in opposite directions with velocities which are as 7 to 2, impinge directly; after impact one of the balls comes to rest, show that this is the lighter ball, and that the heavier ball moved with the less velocity. What also is the modulus of elasticity? Ans. $\frac{4}{45}$.

8. If the momenta of two bodies before impact are equal and opposite, show that they are also equal after impact.

9. An inelastic ball of m lbs. moving vertically upwards with a velocity mg, impinges upon another of n lbs. moving downwards with a velocity ng; m being greater than n, through what height will the balls rise after impact? Ans. $\frac{1}{2}(m-n)^2g$.

10. If in the preceding the balls be elastic, what interval will elapse between the return of the two balls to the point of impact? Ans. $2e(m+n)$ seconds.

11. If an elastic ball fall vertically through a height h on to a fixed horizontal plane, to what height will it rise after its first rebound? Ans. e^2h.

12. An elastic ball is projected vertically upwards from a horizontal plane with a velocity V; after what interval will it cease to rebound? Ans. $\frac{2V}{g(1-e)}$.

13. An elastic ball strikes against two smooth vertical planes at right angles to each other; show that if the motion of the ball be in a horizontal plane, the direction of the ball after the second impact is parallel to its direction before the first impact.

4. A smooth ball moving with a velocity V impinges upon ther ball at rest, at an angle of 30° to the common tangent; mass of the second ball is double that of the first, and the lulus of elasticity is $\frac{1}{2}$; what is the velocity of the larger ball r impact? Ans. $\frac{1}{4}$V.

5. In the preceding, show that the two balls after impact move lirections at right angles to each other.

6. If two smooth spherical balls impinge obliquely, show that balls will not both move after impact in the direction of the mon tangent unless $e = 0$.

7. A smooth spherical ball impinges upon another moving with qual velocity in a direction at right angles to its own, but on opposite side of the common tangent; show that if the mass he impinging ball be e times that of the other, the former impact will move in a direction inclined at 135° to the com- tangent.

8. A smooth spherical ball whose mass is 9 impinges at an e whose sine is $\frac{1}{2}$ upon another ball whose mass is 21, moving an equal velocity, at an angle whose sine is $\frac{1}{3}$, on the other of the common tangent; the modulus of elasticity is $\frac{1}{2}$; show after impact the balls have equal velocities.

9. What is the condition that two balls moving with equal cities before impact in directions that fall upon opposite sides the common tangent, shall have equal velocities after impact?

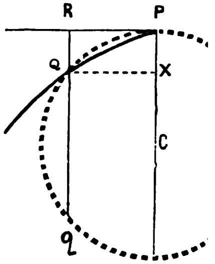
Ans. $$e^2(M - M')(\sin\alpha + \sin\alpha') + 2e(M\sin\alpha - M'\sin\alpha') + (M + M')(\sin\alpha - \sin\alpha') = 0.$$

CHAPTER XIII.

ON THE FREE CURVILINEAR MOTION OF A PARTICLE, AND MOTION ABOUT CENTRES OF FORCE.

223. By the first law of motion, a body in motion, unacte by any force, will move on in a straight line. Curvilinear can therefore result only from the continued action of one o forces, and these forces are either partly or wholly emplo overcoming at each instant the tendency of the body to rect motion. Let a single force be conceived of as producing, *respect only*, the same effect as the force or forces really actin the body. The direction of this force must at each instant right angles with the tangent to the path in which the b moving. A force equal and opposite to this is taken to re the tendency of the body to rectilinear motion, and is call *centrifugal force.*

224. *To find the centrifugal force when a body is movin curvilinear orbit with a velocity* v.



Let the body when moving with the velocity v be at the point P, and let PR be the tangent and PC the normal at P. The force which deflects the body at P acts in the direction of the normal PC, and, by definition, the centrifugal force is a force equal and opposite to this. Let Q be the position of the body after a short interval t, and draw QR parallel to PC, then, if the force be supposed to be u during the interval t, the second law of motion is applicab

Art. 150), and PR is the space due to the velocity at P, and PX or QR the space due to the force. Then, if f be acceleration due to the deflecting force

$$\mathrm{PR} = vt,$$

and $$\mathrm{QR} = \tfrac{1}{2}ft^2,$$

whence $$f = \frac{2\,\mathrm{QR}\,v^2}{\mathrm{PR}^2}.$$

The error arising from the supposed uniformity of the force is diminished the nearer Q is taken to P; hence, ultimately,

$$f = \text{limit of}\,.\,\frac{2\,\mathrm{QR}\,v^2}{\mathrm{PR}^2}.$$

To determine the value of this limit, let the circle PQq be drawn passing through the points P, Q, and touching the line PR in P. Let RQ produced meet the circle in q. Then, *Euclid*, iii. 36,

$$\mathrm{PR}^2 = \mathrm{RQ}\,.\,\mathrm{R}q,$$

$$\therefore \qquad \frac{\mathrm{RQ}}{\mathrm{PR}^2} = \frac{1}{\mathrm{R}q}.$$

But, as Q approaches P, Rq becomes more and more nearly equal to the diameter of the circle, and the circle itself to the circle of curvature. (See Appendix.) Hence, if ρ be the radius of curvature,

$$\text{limit of } \frac{\mathrm{RQ}}{\mathrm{PR}^2} = \text{limit of } \frac{1}{\mathrm{R}q} = \frac{1}{2\rho},$$

and $\therefore$ $$f = \frac{v^2}{\rho};$$

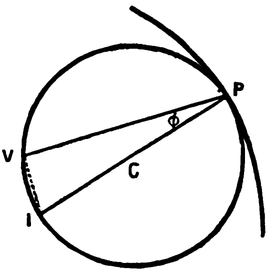
or the centrifugal force is a force which produces an acceleration equal to $\frac{v^2}{\rho}$.

Cor. 1. Hence, if a particle move freely in a curvilinear orbit under the action of any force, and if F be the acceleration due to the force at any point P, and ϕ be the angle between the direction of the force and the normal at P, then

$$\mathrm{F}\cos\phi = \frac{v^2}{\rho}.$$

or, $$\mathrm{F} = \frac{v^2}{\rho\cos\phi}.$$

COR. 2. Let PV be the chord of curvature, (*i.e.* the chord of the circle of curvature,) drawn through P in the direction of the force, then $PV = PI \cos\phi = 2\rho\cos\phi$, and hence by the preceding corollary,

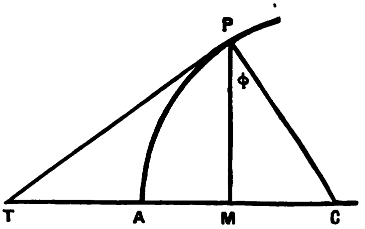


$$v^2 = F \cdot \frac{PV}{2} = 2F \cdot \frac{PV}{4}.$$

Comparing this with the formula, $v^2 = 2fs$, Art. 169, we see that when a body moves in a curvilinear orbit under the action of any force, the velocity at any point is that which would be acquired by a body falling through one-fourth of the chord of curvature drawn in the direction of the force, under the action of the force at that point supposed to be uniform.

225. *To determine the law of force when a particle moves in a circular orbit under the action of a force at right angles to one of the diameters.*

Let AM be the given diameter, P any position of the particle, and F the acceleration due to the force at P. Then, if G be the centre of the circle, and r its radius,



$$F = \frac{v^2}{r\cos\phi}.$$

But, by hypothesis, the force always acts in a direction at right angles to AM, it can therefore produce no acceleration in the direction of AM, and hence the velocity of the particle in this direction must be constant. Let V represent this constant velocity, and through P draw the tangent PT, then PT is the direction of the velocity v, and $v \cos PTM$ is its component in the direction of AM, therefore,

$$V = v\cos PTM,$$
$$= v\cos\phi;$$

whence,
$$F = \frac{V^2}{r\cos^3\phi},$$
$$= \frac{V^2 r^2}{r^3 \cos^3\phi},$$
$$= \frac{V^2 r^2}{PM^3},$$

∴
$$F \propto \frac{1}{PM^3},$$

or the acceleration and therefore the force varies inversely as the cube of the distance from the diameter AG.

Cor. Since $v = \dfrac{V}{\cos\phi}$ and $\cos\phi = \dfrac{PM}{r}$,

we have
$$v = \frac{Vr}{PM},$$

∴
$$v \propto \frac{1}{PM};$$

or the velocity varies inversely as the distance of the particle from the diameter AG.

226. *To determine the law of force when a particle moves in a parabolic orbit under the action of a force at right angles to the axis.*

In the fig. of preceding Art. let AM be the axis of the parabola, PG the normal, and PT the tangent at P, then as before,
$$F = \frac{v^2}{\rho\cos\phi},$$

and
$$v = \frac{V}{\cos\phi},$$

whence
$$F = \frac{V^2}{\rho\cos^3\phi}.$$

Let $4c$ be the *latus rectum* of the parabola, and let r be the focal distance of P; then (see Appendix)
$$\rho = 2c\left(\frac{r}{c}\right)^{\frac{3}{2}}$$

and $$\cos\phi = \frac{\text{PM}}{2\sqrt{(cr)}},$$

$\therefore$ $$\rho\cos^3\phi = \frac{\text{PM}^3}{4c^2},$$

and $$\text{F} = \frac{4\text{V}^2c^2}{\text{PM}^3};$$

or $$\text{F} \propto \frac{1}{\text{PM}^3}.$$

COR. Sin $v = \dfrac{\text{V}}{\cos\phi}$, and $\cos\phi = \dfrac{\text{PM}}{2\sqrt{(cr)}}$,

$\therefore$ $$v = \frac{2\text{V}\sqrt{(cr)}}{\text{PM}}.$$

227. *To determine the law of force when a particle moves elliptic orbit under the action of a force at right angles to the axis.*

Using the same notation as in the preceding, we have as

$$\text{F} = \frac{\text{V}^2}{\rho\cos^3\phi}.$$

Let a and b be the semi-axis of the ellipse, and r, r' the distances of P, then (see Appendix)

$$\rho = \frac{(rr')^{\frac{3}{2}}}{ab},$$

and $$\cos\phi = \frac{a}{b}\cdot\frac{\text{PM}}{\sqrt{(rr')}}.$$

Therefore, $$\text{F} = \frac{\text{V}^2\,b^4}{a^2\,\text{PM}^3},$$

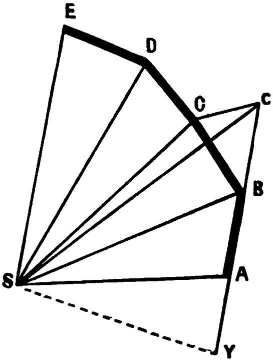
or $$\text{F} \propto \frac{1}{\text{PM}^3}.$$

COR. Since $v = \dfrac{\text{V}}{\cos\phi}$, and $\cos\phi = \dfrac{a}{b}\cdot\dfrac{\text{PM}}{\sqrt{(rr')}}$;

$$v = \frac{\text{V}b\sqrt{(rr')}}{a\text{PM}}.$$

228. A *central force* is a force which always tends towards a fixed point or centre. Central forces, when attractive, are sometimes called *centripetal* forces.

229. "*If a body move in any orbit about a fixed centre of force, the areas described by lines drawn from the centre to the body lie in one plane, and are proportional to the times of describing them.*" (Newton, *Principia;* book i. sect. 2, prop. i.)



Let S be the centre of force; and suppose a body unattracted by the force in S to describe the straight line AB with a uniform velocity in τ''. Then, if suffered to proceed, it would move on uniformly in the direction of AB produced, and describe $Bc = AB$ in the next interval of τ''; but at B suppose an instantaneous impulse communicated to it in the direction BS, which causes it to move in the direction BC; then, by the second law of motion, if Cc be drawn parallel to BS, the body at the end of the second interval will be found at C. Draw SA, SB, SC, Sc. Since Cc is parallel to BS, the triangles SBC, SBc, are equal; and since $Bc = AB$, the triangles SBc, SAB are equal. Therefore the triangles SBC, SAB are equal, and are in the same plane, since no force has acted to draw the body out of the plane SAB. Similarly, if impulses be communicated at the end of every interval of τ'', in directions tending always to S, causing the body to describe CD, DE, &c., in the third, fourth, &c., intervals, the triangles SCD, SDE, &c., will be each equal to SAB, and in the same plane with it; also their bases AB, BC, CD, &c., are described in equal times. Therefore, the area of any number of these triangles, or the polygon SABCDE, varies as the time of describing it. Now let the number of intervals be increased, and the magnitude of each diminished indefinitely, then the polygon approximates to a curvilinear area, the sum of the impulses to a continued force always tending to S as their limits, and what was proved of those quantities is true of their limits, and therefore the curvilinear area described in any time is proportional to the time.

COR. 1. If v be the velocity of the body at A, and p the perpendicular from S upon the tangent at A, then

$$\text{area described in } t'' = \tfrac{1}{2} ptv.$$

For, draw SY perpendicular to AB, then since AB is ultimately the tangent, SY is ultimately equal to p. Let $t = n\tau$, and, as before, let AB be the space described in the interval τ with the velocity v, then $\text{AB} = v\tau$, and

$$\begin{aligned}\text{area described in } \tau'' &= \tfrac{1}{2}\,\text{SY} \times \text{AB},\\ &= \tfrac{1}{2}\,\text{SY} \times v\tau.\end{aligned}$$

But, by the preceding proposition, the area described in t'' is n times the area described in τ''; therefore,

$$\begin{aligned}\text{area described in } t'' &= \tfrac{1}{2}\,\text{SY} \times vn\tau,\\ &= \tfrac{1}{2}\,\text{SY} \times vt,\\ &= \tfrac{1}{2}\,pvt \text{ ultimately.}\end{aligned}$$

COR 2. If h be equal to twice the area described in $1''$, then, by preceding corollary, $$h = pv,$$
and hence $$v = \frac{h}{p},$$
or the velocity at any point varies inversely as the perpendicular upon the tangent at that point.

230. "*If a body moving in a plane curve describe areas proportional to the times by lines drawn from the body to any point, the body is acted on by forces all tending to that point.*" (Newton, *Principia;* book i. sect. 2, prop. ii.)

Let S (fig. Art. 229) be the point about which areas proportional to the times are described, and suppose, as in the preceding proposition, that a body, unacted upon by any force, describes the straight line AB in τ''. In AB produced, take $\text{B}c = \text{AB}$, then, if suffered to proceed, the body would be at c at the end of the next interval of τ''; but at B suppose an impulse communicated which causes it to describe BC in the second interval, such that the triangle SBC is equal to and in the same plane with the triangle SAB. Join C, c and S, c. Then, since the triangles SBc, SBC are each equal to the triangle SAB, they are equal to each other, and therefore Cc is parallel to SB, and therefore, by the second law of motion, the impulse communicated at B is in the direction BS. Similarly, if D, E, &c., be the places of the body at the ends of the third, fourth, &c., intervals of τ'', so that the triangles SAB, SBC, SCD, &c., are all equal, all the impulses communicated may be shown in like manner to tend to S.

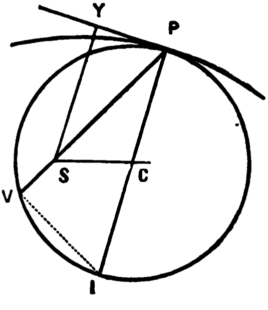
Now, suppose the number of intervals increased, and the magnitude of each diminished indefinitely, then the limit of the polygon is the curvilinear area, and that of the sum of the impulses a continued force tending to S; therefore the body is acted upon by a force tending to S.

231. *If a body move in any orbit about a centre of force* (S), *and if* ρ *be the radius of curvature at any point* (P), p *the perpendicular from the centre upon the tangent at the given point, and* h *twice the area described in* 1'', *then if* F *be the acceleration due to the central force,*

$$F = \frac{h^2\,SP}{p^3\rho}.$$

For, let PG be the normal, and PY the tangent at P, let SY be perpendicular to PY, and the angle SPG $= \phi$. Then, by Art. 224, Cor 1,

$$F = \frac{v^2}{\rho\cos\phi}.$$



But (Art. 228, Cor. 2) $v = \dfrac{h}{p}$; therefore,

$$F = \frac{h^2}{p^2\rho\cos\phi}$$

$$= \frac{h^2\,SP}{p^2\rho\,SP\cos\phi},$$

$$= \frac{h^2\,SP}{p^3\rho}, \text{ since } SP\cos\phi = p.$$

By aid of this expression, the law of force can be readily found, when the orbit in which a body is moving about a centre of force is given. The following sections are examples of its application to such problems.

232. *To find the law of force when a body moves in a circle under the action of a force tending to a given point.*

Let S (fig. Art. 231) be the given point, and PVI the given circle whose radius PG $= a$. Then, in this case, $\rho = a$, and therefore

$$F = \frac{h^2 \, SP}{p^3 \, a}.$$

Let PS, or PS produced, meet the circle in V; then, since SP $\cos \phi = p$, and $2a \cos \phi = PV$,

$$p = \frac{SP \, . \, PV}{2a},$$

whence $$F = \frac{8h^2a^2}{SP^2 \, . \, PV^3};$$

or, $$F \propto \frac{1}{SP^2 \, . \, PV^3}.$$

233. *To find the law of force when a body moves in an ellipse under the action of a force tending to the centre.*

If a, b are the semi-axes of the ellipse, and r, r' the focal distances of any point, then (see Appendix)

$$\rho = \frac{(rr')^{\frac{3}{2}}}{ab},$$

and since S is at the centre,

$$p = \frac{ab}{\surd(rr')},$$

$$\therefore \quad p^3\rho = a^2b^2,$$

$$\therefore \quad F = \frac{h^2 \, . \, SP}{a^2b^2};$$

or, $$F \propto SP,$$

that is, the force varies directly as the distance.

Cor. 1. Hence, if μ be the value of F at the unit of distance, or, as it is termed, the absolute force, then

$$F = \mu \, . \, SP,$$

and
$$\mu = \frac{h^2}{a^2 b^2}.$$

Cor. 2. Let P be the periodic time, or the time of describing the complete ellipse, then, by Art. 229,

$$\frac{P}{1''} = \frac{\text{area of ellipse}}{\text{space described in } 1''},$$

$$P = \frac{2\pi ab}{h};$$

but if μ be the absolute force, $h = ab\sqrt{\mu}$, therefore,

$$P = \frac{2\pi}{\sqrt{\mu}},$$

or the periodic time is independent of the dimensions of the ellipse, and hence, if several bodies move in different ellipses about the same force in the centre, the times of their revolutions will be the same.

234. *To find the law of force when a body moves in an ellipse under the action of a force tending towards one of the foci.*

Let a and b be the semi-axes of the ellipse, r and r' the focal distances of any point, the former being measured from that focus towards which the force is directed; then (see Appendix)

$$p = b\sqrt{\left(\frac{r}{r'}\right)},$$

and as before,
$$\rho = \frac{(rr')^{\frac{3}{2}}}{ab},$$

$$\therefore \quad p^3 \rho = \frac{b^2 r^3}{a} = \frac{b^2 . SP^3}{a},$$

$$\therefore \quad F = \frac{ah^2}{b^2} \cdot \frac{1}{SP^2},$$

or,
$$F \propto \frac{1}{SP^2}.$$

Cor. If P be the periodic time, then, as before,

$$P = \frac{2\pi ab}{h};$$

but if μ be the absolute force, $\mu = \frac{ah^2}{b^2}$; and, therefore,

$$P = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

Hence, if several bodies move in different ellipses, abou same force in the focus, the squares of the periodic times a the cubes of the major axes.

235. The values of p and ρ being the same for the hyperb for the ellipse, it follows that the investigations of Arts. 232 will serve also for the cases of a body moving in a hyperbolic about a force directed severally to the centre and the focus.

236. *To find the law of force when a body moves in a parc orbit about a force tending towards the focus.*

Let $4c$ be the *latus rectum* of the parabola, and r the focal tance of any point, then (Appendix)

$$\rho = 2c\left(\frac{r}{c}\right)^{\frac{3}{2}},$$

and $$p = \sqrt{(cr)},$$

$\therefore$ $$p^3\rho = 2cr^3 = 2c \, . \, SP^3.$$

$\therefore$ $$F = \frac{h^2}{2c} \cdot \frac{1}{SP^2};$$

or, $$F \propto \frac{1}{SP^2}.$$

Cor. Since $v = \frac{h}{p}$, and $p = \sqrt{(cr)}$, therefore,

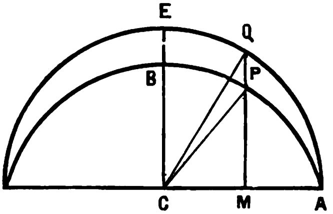
$$v = \frac{h}{\sqrt{(cr)}},$$

or, $$v^2 \propto \frac{1}{r},$$

that is, the square of the velocity varies inversely as the dista

The following problem is of especial importance in the theory of ndulations.

237. *A body moves in an elliptic orbit under the action of a force directed towards the centre, to find the position of the body at any time* t.



Let APB be the ellipse, P the position of the body at the time t, and, for greater simplicity, suppose the time to be reckoned from the instant when the body was at B. Then (Art. 229),

$$\frac{t}{1''} = \frac{2 \text{ area BCP}}{h}.$$

With C as a centre, and CA as a radius, describe the circle AQE, and let the ordinate through P meet this circle in Q. Let a and b be the semi-axis of the ellipse, and x the abscissa of the point P. Then,

$$\begin{aligned}\text{area BCMP} &= \frac{b}{a} \text{ area ECMQ},\\ &= \frac{b}{a}(\text{CMQ} + \text{ECQ}),\\ &= \frac{b}{a}\left(\frac{x \,.\, \text{MQ}}{2} + \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2}\right),\\ &= \frac{x \,.\, \text{MP}}{2} + \frac{ab \sin^{-1}\left(\frac{x}{a}\right)}{2},\\ &= \text{area CMP} + \frac{ab \sin^{-1}\left(\frac{x}{a}\right)}{2}.\end{aligned}$$

But $\qquad \text{area BCP} = \text{area BCMP} - \text{area CMP},$

$\therefore \qquad = \dfrac{ab \sin^{-1}\left(\frac{x}{a}\right)}{2}.$

Therefore, $\qquad t = \dfrac{ab \sin^{-1}\left(\frac{x}{a}\right)}{h}.$

But if μ be the absolute force, then (Art. 232, Cor. 1) $h = ab\sqrt{\mu}$
therefore

$$t = \frac{\sin^{-1}\left(\frac{x}{a}\right)}{\sqrt{\mu}},$$

and hence

$$x = a \sin \sqrt{\mu} \,.\, t.$$

Cor. As this result is independent of the minor axis of the ellipse, it remains true, whatever value we may give to b. We may, therefore, suppose b to vanish, and we have the case of a body moving in a straight line, under the action of a force varying directly as the distance.

EXAMPLES.

1. To find the law of force when a particle moves in a parabolic orbit under the action of a force parallel to the axis.

The force is a constant force.

2. To find the law of force when a particle moves in a hyperbolic orbit under the action of a force parallel to the major axis.

3. To find the law of force when a particle moves in a parabolic orbit under the action of a force parallel to a given straight line.

Draw a tangent to the parabola parallel to the given line, and let y' be the ordinate of the particle referred to this tangent, and the diameter through the point of contact as axes, then

$$F \propto \frac{1}{y'^3}.$$

4. Several bodies move in parabolic orbits under the action of the same force in the focus, show that the square of the time of moving from the vertex to the extremity of the *latus rectum* varies as the cube of the *latus rectum*.

5. A body moves in a parabola under the action of a force directed to a given point in the axis produced, find the law of force.

Let S be the centre of force, A the vertex of the parabola, P any position of the body, and M the point where the ordinate through P cuts the axis, then

$$F \propto \frac{SP}{(2SA - SM)^3}.$$

6. A body moves in a parabola about a force in the focus, show that the centre of the circle which passes through the body, the focus, and the vertex, moves uniformly with a velocity equal to 3·8ths of the velocity of the body at the vertex. (Newton, *Princip.* i. prop. xxx.)

7. A body moves in an ellipse under the action of a force directed towards one of the extremities of the major axis, find the law of force.

Let S, the centre of force, be the origin, and SM the abscissa of the point P, then

$$F \propto \frac{SP}{SM^3}.$$

8. A body moves in a circle under the action of a force directed to a point (S) in the circumference, A is the other extremity of the diameter through S, find the time of moving from A to S.

Let μ be the absolute force, and a the radius of the circle, then the time required is

$$\frac{2\pi a^3\sqrt{2}}{\sqrt{\mu}}.$$

9. Two particles move in circular orbits about forces in their centres, the absolute force and the magnitude of the orbits is the same in both cases, but in the one case the force varies as the inverse square of the distance, and in the other as the inverse cube of the distance, compare their periodic times.

Let a be the radius of the orbits, T_1 the periodic time in the first case, and T_2 in the second, then

$$T_2^2 = aT_1^2.$$

CHAPTER XIV.

ON CONSTRAINED MOTION.

238. *If a body fall down a smooth curve under the action of gravity, its velocity at any point is that due to the vertical height through which it has fallen.*

It has already been shown, that the velocity acquired by any body in falling down a smooth inclined plane is that due to the vertical height through which it has fallen. Hence, if a body fall in succession down a series of inclined planes, the velocity acquired will be that due to the total height through which it has fallen, provided that there be no change of velocity in passing from one plane to another. A smooth curve may be regarded as made up of a series of indefinitely small inclined planes, whose reaction, being always at right angles to the direction of the motion, produces no change in the velocity of a body moving over them. Consequently, the velocity at any point is that due to the vertical height through which the body has fallen.

239. *To find the reaction of the curve when a particle moves over a fixed curve under the action of any force.*

Let N be the resolved part of the given force in the direction of the normal, and R the reaction at any point. Then the three forces N, R, and the centrifugal force, must be in equilibrium; and, since they act along the same line, their algebraic sum must be equal to zero. Let the direction of the centrifugal force be taken as the positive direction, then R is positive when the particle moves over the convex side of the curve, and negative when the particle moves over the concave side; hence

$$\pm R + N + \frac{mv^2}{\rho} = 0,$$

he forces be measured in units of force, or

$$\pm R + N + \frac{Wv^2}{g\rho} = o,$$

the forces be measured by pounds.

In applying this formula, the student must remember that N is ;ative when non-concurrent with the centrifugal force.

In a similar way the tension in rods or cords may be found, en a body moving in a curve is connected with a fixed point a rod or cord.

The following are examples in illustration:

Ex. 1. A body, whose weight is W, is connected with a fixed nt by an inextensible cord or rod whose length is r, supposed e without weight, and moves with a uniform velocity v about point as a centre; to find the tension in the cord or rod.

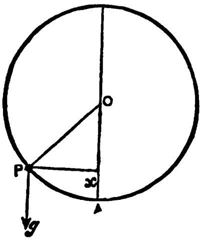
Iere the centrifugal force and the tension are the only forces ng on the rod or cord;

$$\therefore \qquad \text{tension required} = \frac{Wv^2}{gr}.$$

x. 2. A body connected with a fixed centre by an inextensible supposed to be without weight, moves over the arc of a e under the action of gravity; to find the tension in the rod ıy point.

et W be the weight of the body, r the length of the rod, h the ht above the lowest point A of the point which the body fell, and x the height e A when in any position P.

v be the velocity of the body at P, then



$$v^2 = 2g(h-x);\ \text{and}\ N = \frac{W(r-x)}{r},$$

$$\therefore \qquad R = \frac{W(r-x)}{r} + \frac{2W(h-x)}{r},$$

$$= \frac{W(r+2h-3x)}{r}.$$

The tension in the rod will be zero when $x = \frac{1}{3}(r + 2h)$; an the body fall from the highest point of the circle, the tensio zero when $x = \frac{5r}{3}$.

Ex. 3. A particle moves from rest down the convex side c circle whose plane is vertical, from a given point in its circ ference, to find where it will leave the curve.

Let A be the highest point of the circle, B the point from wh the particle starts, and P the position of the particle at any insta Let α and θ be the angles made with the normal through A by t normals through B and P. Then if R be the reaction at P, r t radius of the circle, and m the mass of the particle,

$$R = mg \cos \theta - m\frac{v^2}{r};$$

but, Art. 238, $$v^2 = 2g(r \cos \alpha - r \cos \theta).$$

$$\therefore \quad R = mg(3 \cos \theta - 2 \cos \alpha).$$

Hence, R is equal to zero, or the particle will leave the cur when

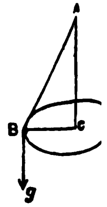
$$\cos \theta = \tfrac{2}{3} \cos \alpha.$$

Or, if h be the height of B from the horizontal diameter, and the height of P when the particle leaves the curve,

$$x = r \cos \theta,$$
$$= \tfrac{2}{3} \cos \alpha = \tfrac{2}{3}h.$$

Ex. 4. A body fastened by an inextensible cord to a fixed poi moves uniformly in a horizontal circle; to find the velocity at a point.

Let m be the mass of the body, and c the length of the co Let B be any position of the body, C the centre of the horizontal circle, and the angle BAC $= \alpha$. In any position the body is in equilibrium under three forces; the centrifugal force, the force of gravity, and the tension in the cord. Therefore, by triangle of forces,



$$mg : \frac{mv^2}{BC} :: \sin ABC : \sin BAC.$$

$BC = c \sin \alpha$, $\sin ABC = \cos \alpha$, and $\sin BAC = \sin \alpha$;

$$\therefore \qquad v^2 = \frac{gc \sin^2 \alpha}{\cos \alpha}.$$

OR. If T be the time of describing an entire revolution, since e = velocity × time,

$$T = \frac{2\pi c \sin \alpha}{v},$$

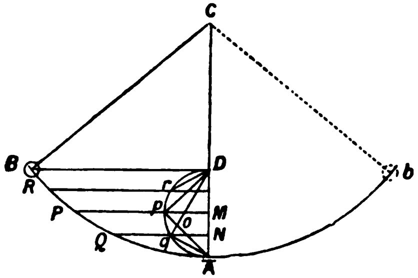
$$= 2\pi \sqrt{\left(\frac{c \cos \alpha}{g}\right)} = 2\pi \sqrt{\left(\frac{AC}{g}\right)}.$$

e the time of a complete revolution is independent of the h of the cord, and depends only upon the height of the fixed above the plane of the circle.

0. A heavy particle attached to a cord or wire whose weight mparison with that of the particle is so small that it may be arded, and so suspended that it may vibrate in arcs of a curve, led a SIMPLE PENDULUM.

1. *To find the time of the vibration of a simple pendulum gh a small circular arc, the resistance of the air being dis-ded.*

t BC be a simple pendulum, whose length is r, and which tes through a small ar arc Bb, having west point at A. P be any point in and let PQ be the ice described in an nitely small portion ne. Draw the ver- radius CA, and the ntal lines BD, PM,



On AD describe the semi-circle $AqpD$, cutting PM and QN nd q. Draw the chords Dp, Dq, Ap, Aq.

The velocity of the pendulum at P is, by Art. 237, $\surd(2g$ and since PQ is indefinitely small the velocity in it may be sidered uniform, and hence if t' be the time of describing PQ

$$t' = \frac{PQ}{\surd(2g\mathrm{DM})}.$$

Since the arc BA is by hypothesis small, any portion of th may be taken as equal to its own chord. Hence the arc chord $PA = \surd(2r\mathrm{AM})$. Similarly arc $QA = \surd(2r\mathrm{AN})$. Whe since $PQ = PA - QA$,

$$t' = \sqrt{\left(\frac{r}{g}\right)} \cdot \frac{\surd(\mathrm{AM}) - \surd(\mathrm{AN})}{\surd(\mathrm{DM})},$$

$$= \sqrt{\left(\frac{r}{g}\right)} \cdot \frac{\surd(\mathrm{AM}\,.\,\mathrm{AD}) - \surd(\mathrm{AN}\,.\,\mathrm{AD})}{\surd(\mathrm{DM}\,.\,\mathrm{AD}}\,;$$

$$= \sqrt{\left(\frac{r}{g}\right)} \cdot \frac{Ap - Aq}{Dp}.$$

But since Q is indefinitely near to P, Ao may be regar equal to Aq, and therefore $Ap - Aq = op$. Hence,

$$t' = \sqrt{\left(\frac{r}{g}\right)} \cdot \frac{op}{Dp} = \sqrt{\left(\frac{r}{g}\right)} \times \text{angle } pDq.$$

Similarly,

$$\text{time of describing RP} = \sqrt{\left(\frac{r}{g}\right)} \times \text{angle } rDp.$$

Hence if the arc BA be divided into indefinitely small arcs, RP, PQ, &c., then, since the time of describing the whole equal to the sum of describing the several parts,

$$\text{time of describing BA} = \sqrt{\left(\frac{r}{g}\right)} \times \text{sum of angle}$$

Now the sum of the angles at D is $\frac{\pi}{2}$. Hence

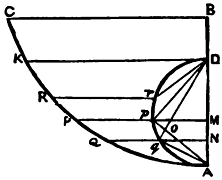
$$\text{time of describing BA} = \frac{\pi}{2}\sqrt{\left(\frac{r}{g}\right)}.$$

But the time of rising through the arc Ab is equal to falling through the arc BA. Hence

$$\text{time of vibration} = \pi\sqrt{\left(\frac{r}{g}\right)}.$$

This result, which applies to pendulums vibrating in a circle only when the arc of vibration is *small,* applies to cycloidal pendulums, however large the arc of vibration may be. This interesting property of the cycloid is deduced in the following sections:

242. *To find the time in which a body falls down an arc of cycloid whose axis is vertical.*



Let the body fall from K down KA, the rc of a cycloid whose axis AB is vertical. et P be any point in KA, and let PQ be e distance described in an indefinitely nall portion of time. Draw the horizon-d lines KD, PM, QN. On AD describe e semi-circle A$qp$D, cutting PM and QN in p and q. Draw e chords Dp, Dq, Ap, Aq. The velocity of the body at P $\sqrt{(2g\text{DM})}$, and since PQ is indefinitely small, the velocity in it ay be considered uniform; therefore, if t' be the time of de-ribing PQ,

$$t' = \frac{\text{PQ}}{\sqrt{(2g\text{DM})}}.$$

It is shown in works that discuss the properties of the cycloid, at if AP be any arc of a cycloid, measured from the lowest int; then $\text{AP}^2 = 8a\,.\,\text{AM}$, where a is the radius of the generating rcle, or the semi-axis of the cycloid. Let $\text{AB} = 2a$, then $\text{PQ} =$ $\text{P} - \text{AQ} = \sqrt{(8a\text{AM})} - \sqrt{(8a\text{AN})}$.

hence,
$$t' = \sqrt{\left(\frac{4a}{g}\right)}\,.\,\frac{\sqrt{(\text{AM})} - \sqrt{(\text{AN})}}{\sqrt{(\text{DM})}},$$
$$= \sqrt{\left(\frac{4a}{g}\right)}\,.\,\frac{\sqrt{(\text{AM}\,.\,\text{AD})} - \sqrt{(\text{AN}\,.\,\text{AD})}}{\sqrt{(\text{DM}\,.\,\text{AD})}},$$
$$= \sqrt{\left(\frac{4a}{g}\right)}\,.\,\frac{\text{A}p - \text{A}q}{\text{D}p}.$$

ut since Q is indefinitely near to P, Ao may be regarded as al to Aq, and therefore $\text{A}p - \text{A}q = op$, and

$$t' = \sqrt{\left(\frac{4a}{g}\right)}.\,\frac{op}{\text{D}p}, = \sqrt{\left(\frac{4a}{g}\right)} \times \text{angle } p\text{D}q.$$

Similarly,

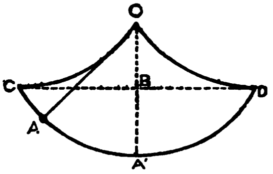
$$\text{time of describing RP} = \sqrt{\left(\frac{4a}{g}\right)} \times \text{angle } rDp.$$

$$\therefore \text{ time of describing KA} = \sqrt{\left(\frac{4a}{g}\right)} \times \text{sum of angles at D.}$$

$$= \frac{\pi}{2}\sqrt{\left(\frac{4a}{g}\right)} = \pi\sqrt{\left(\frac{a}{g}\right)}.$$

As this result is independent of the length of the arc KA, it shows that the time of falling to the lowest point of a cycloid down any arc is the same in all. From this property, the cycloid is called a *tautochronous* curve.

243. If OC, OD be two equal semi-cycloids, whose axis is $2a$, and a cord OA, equal in length to OC or OD, be fastened at O, and wrapped round OC or OD, the point A will, as the cord is unwrapped from the one semi-cycloid and wrapped around the other, describe a cycloid CAD, whose axis also is $2a$. Hence, if OC and OD be placed in a vertical plane, and a cord whose length is $4a$ be fastened at O, a body suspended at A will oscillate in cycloidal arcs, and consequently form a pendulum whose oscillations are equal in time, whatever their extent. If l be the length of such a pendulum, and t the time of an oscillation,



$$t = 2\pi\sqrt{\left(\frac{l}{4g}\right)},$$

$$= \pi\sqrt{\left(\frac{l}{g}\right)}.$$

244. It will be seen from the foregoing results, that when the force of gravity remains the same, the time of the oscillation of a simple pendulum varies directly as the square root of the length, and that when the length is the same, the time varies inversely as the square root of the force of gravity.

245. *To find the number of oscillations gained or lost in a given period, when the length of a pendulum is slightly altered.*

Let l be the length of a pendulum making n oscillations in the time T. Let the length of the pendulum become $l+x$ (x being a very small quantity), and let n' be the number of oscillations then made in the time T. Then T being measured in seconds,

$$\frac{T}{n} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}, \text{ and } \frac{T}{n'} = \pi\left(\frac{l+x}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \frac{n'}{n} = \left(\frac{l}{l+x}\right)^{\frac{1}{2}} = \left(1+\frac{x}{l}\right)^{-\frac{1}{2}};$$

$$\therefore \quad \frac{n-n'}{n} = 1 - \left(1+\frac{x}{l}\right)^{-\frac{1}{2}}, = \frac{x}{2l} \text{ nearly};$$

$$\therefore \quad n-n' = \frac{nx}{2l}.$$

246. *To find the number of oscillations gained or lost in a given period, when the force of gravity is slightly altered.*

Let a pendulum, whose length is l, make n oscillations in a time T, when the force of gravity is g, and n' when the force of gravity is $g+x$. Then,

$$\frac{T}{n} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}, \text{ and } \frac{T}{n'} = \pi\left(\frac{l}{g+x}\right)^{\frac{1}{2}};$$

$$\frac{n'}{n} = \left(1+\frac{x}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \frac{n'-n}{n} = \left(1+\frac{x}{g}\right)^{\frac{1}{2}} - 1 = \frac{x}{2g} \text{ nearly};$$

$$\therefore \quad n'-n = \frac{nx}{2g}.$$

247. *To find the height of a station above the earth by observing the number of oscillations lost by a pendulum in a given period, the earth being supposed to be spherical.*

Let a pendulum, whose length is l, make n oscillations in the time T, under the action of gravity, at the surface of the earth.

Let the same pendulum make n' oscillations in the time T, when at a height h above the surface of the earth; let r be the radius of the earth.

The force of gravity varies inversely as the square of the distance from the centre of the earth. Therefore, if G be the force of gravity at the height h,

$$G : g :: \frac{1}{(r+h)^2} : \frac{1}{r^2},$$

or,

$$G = \frac{gr^2}{(r+h)^2}.$$

But $n : n' :: \sqrt{g} : \sqrt{G}$; therefore,

$$\frac{n}{n'} = \frac{r+h}{r};$$

$$\therefore \quad h = \frac{(n-n')\,r}{n'} = \frac{(n-n')\,r}{n} \text{ nearly.}$$

248. *To find the depth of a mine by observing the number of oscillations lost by a pendulum in a given period, upon the supposition that the earth is a sphere of uniform density.*

Let G be the force of gravity at the bottom of the mine, and h the depth of the mine. Since the earth is regarded as a sphere of uniform density, the force of gravity in its interior varies directly as the distance from the centre. Therefore,

$$\frac{G}{g} = \frac{r-h}{r},$$

Whence,

$$\frac{n'}{n} = \sqrt{\left(\frac{G}{g}\right)} = \sqrt{\frac{r-h}{r}};$$

$$\therefore \quad \left(\frac{n'}{n}\right)^2 = \frac{r-h}{r},$$

or,

$$\frac{h}{r} = 1 - \left(\frac{n'}{n}\right)^2 = \frac{2\,(n-n')}{n} \text{ nearly.}$$

EXAMPLES.

1. A perforated ball moves over a smooth circular ring whose plane is vertical, the ball is projected upward from one extremity of the horizontal diameter, with such a velocity that it will come to rest at the highest point of the ring, show that the reaction is zero when the ball has moved over an arc subtending an angle whose sine is $\frac{2}{3}$.

2. In the preceding, with what velocity must the ball be projected that the reaction may be zero, when the ball reaches the summit?

The required velocity is equal to $\surd(3gr)$.

3. A circle is placed with its plane vertical, with what velocity must a particle be projected vertically upwards, from one extremity of the horizontal diameter along the interior of the circle, that after quitting the curve it may pass through the centre?

The required velocity is equal to $\surd(gr\surd 3)$.

4. A body moves from rest over a circular quadrant from its highest to its lowest point, the radius at the highest point being horizontal and at the lowest vertical, show that on reaching the lowest point the reaction is equal to three times the force of gravity.

5. In the preceding, if the radius at the highest point be vertical and at the lowest horizontal, the reaction at the lowest point is equal to twice the force of gravity.

6. A particle moves from rest over the convex side of an ellipse whose major axis is vertical, from a given point in the curve, to find when it will leave the ellipse.

Let h be the height of the given point above the minor axis, that of the point at which the particle will leave the ellipse, th the value of x is determined by the equation,

$$(e^2x^3 - 3a^2x + 2a^2h = 0.$$

7. An ellipse is placed with its minor axis vertical, with what velocity must a particle be projected vertically from the extremity of the major axis along the interior of the ellipse, that after quitting the curve it may pass through the centre?

Let a and b be the semi-axes of the ellipse, the velocity required will be equal to

$$\left\{\frac{g(a^2 + 8b^2)}{3b\sqrt{3}}\right\}^{\frac{1}{2}}.$$

CHAPTER XV.

ON WORK.

249. Work is said to be done whenever a body is moved against resistance, as, for instance, when a weight is raised from the ground, or when a cannon-ball pierces an iron plate.

The ordinary unit of work is the amount of work done in raising a weight of one pound vertically through one foot, and is called the *foot pound;* and hence work is measured by the product of the number of *pounds* in the force against which the body is moved and the number of *feet* in the distance through which it is moved, this distance being measured in the direction of the resistance.

The quantity of work done in a unit of time is taken as the measure of the efficiency of any agency by which work is perormed. The unit of time commonly taken for this purpose is one minute. The power of doing 33,000 foot-pounds of work in one minute is termed a horse-power, and is the unit commonly used in measuring the efficiency of steam-engines.

250. The power of doing work possessed by any body is termed ts *energy.* A body in motion possesses energy since it can impel another body against resistance; and a heavy body in a position from which it can fall possesses energy since by its fall it can raise another body against the force of gravity. In the former case the energy is called *kinetic energy,* in the latter, energy of position. The former is also sometimes described as *actual* energy, the latter as *potential* energy.

The energy possessed by any body is often spoken of as the work stored up or accumulated in the body.

251. *To find the work stored up in a body whose weight is* W *moving with a velocity* v.

Let P be the force measured in pounds, which acting through the space s will bring the body to rest, then Ps is the amount of work which the body is capable of doing. The acceleration generated by the force P in the weight W is (Art. 154) $\frac{Pg}{W}$; and hence

$$v^2 = \frac{2Pgs}{W};$$

$$\therefore \qquad Ps = \frac{Wv^2}{2g}, \qquad \text{(i.)}$$

If h be the height through which a body must fall to acquire the velocity v, then $v^2 = 2gh$. Substituting this in the results just obtained we have also

$$Ps = Wh, \qquad \text{(ii.)}$$

or the work stored up is equal to the product of the weight of the moving body, and the height through which it must fall to acquire the velocity with which it is moving.

Also, if M be the mass of the body, then, since $M = \frac{W}{g}$, it follows from (i.) that

$$Ps = \tfrac{1}{2}Mv^2, \qquad \text{(iii.)}$$

or the work stored up is one-half the *vis viva* of the body.

252. When a particle moves uniformly in a circle it is convenient to measure its velocity by the angle subtended at the centre of the circle by the arc traversed in one second. This is termed its *angular velocity.* If r be the radius of the circle, v the linear velocity, and ω the angular velocity given in circular measurement,

$$v = \omega r.$$

If ω is given in degrees, then

$$v = \frac{\pi\omega r}{180}.$$

If a body rotate uniformly about any point in itself, the angular velocity is evidently the same at every part, hence the linear velocity of each particle in the body is the angular velocity multiplied by the distance of the particle from the axis of rotation.

253. *To find the work stored up in a thin rod of uniform thickness and density when rotating uniformly about one of its extremities.*

Let W be the weight of the rod, and l its length. Let the rod be supposed to be divided into n equal parts, n being a number indefinitely large. The weight of each of these parts is consequently $\frac{W}{n}$.

Their distances from the axis of rotation may be taken as

$$\frac{nl}{n}, \frac{(n-1)l}{n}, \frac{(n-2)l}{n}, \text{ \&c., respectively.}$$

If ω be the angular velocity, the linear velocities of these parts will be $\frac{\omega nl}{n}, \frac{\omega(n-1)l}{n}, \frac{\omega(n-2)l}{n}$, &c., respectively.

Then the work done by these parts will be $\frac{W\omega^2n^2l^2}{2gn \,.\, n^2}, \frac{W\omega^2n(-1)^2l^2}{2gn \,.\, n^2}$, &c. Hence the whole work will be

$$\frac{W\omega^2l^2}{2g}\left(\frac{n^2+(n-1)^2+(n-2)^2+\text{\&c.}+1}{n^3}\right).$$

The larger n is the more nearly does this result represent the work stored in the rod. But the larger n is the more nearly does the expression in brackets approach in value to ⅓. (See Art. 377.) Hence the work stored in the rod is

$$\frac{W\omega^2l^2}{6g}.$$

If v be the velocity of that extremity of the rod which is remote from the axis, then $v = \omega l$. Then the work stored in the rod is

$$\frac{Wv^2}{6g} \text{ or } \frac{Wh}{3} \text{ (if } v^2 = 2gh).$$

Comparing these results with those obtained in Art. 251, we see that the work done by a rod rotating about one extremity is one-third of the work which would be done by the same rod if all its parts moved with the velocity of the part most remote from the axis.

254. The investigation of the preceding section applies equally to the case of a thin rod rotating uniformly about its middle point, since this may be regarded as consisting of the equal rods rotating about their extremities with equal angular velocities. The work done by whole rod is the sum of the work done by each of the parts; that is to say, is double the work done by either. As the weight of each part is one-half the weight of the whole rod, it still remains true in this case also that the work done by the rotating rod is one-third of that which would be done by the same rod if all its parts were moving with the velocity of the part most remote from the axis.

255. *To find the work stored up in a circular dish of uniform thickness and density when rotating uniformly about its centre.*

Let W be the weight and r the radius of the disk. Let w be the weight of a square unit of the disk and ω the angular velocity.

Let the disk be supposed to be divided into n concentric rings, each of the breadth $\frac{r}{n}$, n being indefinitely large.

The radius of these rings beginning with the outermost will be

$$\frac{nr}{n}, \frac{(n-1)r}{n}, \frac{(n-2)r}{n}, \text{ \&c., respectively.}$$

The weight of the rings will be *

$$\frac{2\pi nr}{n} \cdot \frac{rw}{n}, \frac{2\pi(n-1)r}{n} \cdot \frac{rw}{n}, \frac{2\pi(n-2)r}{n} \cdot \frac{rw}{n}, \text{ \&c., respectively.}$$

The velocity of the rings will be

$$\frac{nr\omega}{n}, \frac{(n-1)r\omega}{n}, \frac{(n-2)r\omega}{n}, \text{ \&c., respectively.}$$

Hence the work done by each will be

$$\frac{\pi n^3 r^4 \omega^2 w}{gn^4}, \frac{\pi(n-1)^3 r^4 \omega^2 w}{gn^4}, \frac{\pi(n-2)^3 r^4 \omega^2 w}{gn^4}, \text{ \&c., respectively.}$$

---

* The area of a plane ring of small breadth is very nearly the product of the circumference and the breadth. Thus let r be the radius and x the breadth, x being small. Then the area of the ring is the difference of the area of two concentric circles whose radii are r and $r-x$, and hence is equal to $\pi[r^2-(r-x)^2]$; that is, neglecting x^2, is equal to $2\pi rx$.

Hence the work done by the disk will be approximately

$$\frac{\pi r^4\omega^2 w}{g}\left(\frac{n^3+(n-1)^3+(n-2)^3+\&c.+1}{n^4}\right).$$

But the larger n is the more nearly is the expression in brackets equal to $\frac{1}{4}$. (See Art. 377.) Hence the work stored in the disk is

$$\frac{\pi r^4\omega^2 w}{4g}.$$

But $W=\pi r^2 w$, and if v be the velocity of a point on its rim, $v=r\omega$. Hence the work stored in the disk is

$$\frac{Wv^2}{4g},$$

or the work done by a rotating disk is one-half the work that would be done by the same disk if all its parts moved with the velocity of the points most remote from the axis of rotation.

256. *To find the work stored in a wheel of uniform thickness and density when rotating uniformly about its centre.*

Let W be the weight of the wheel, r and r' its outer and inner radii, ω its angular velocity.

The wheel may be regarded as the difference of two concentric disks of the radii r and r' respectively, rotating with the same angular velocity ω. Hence the work done by the wheel is the difference of the work done by these disks; that is, by the preceding article,

$$\begin{aligned}&=\frac{\pi r^4\omega^2 w}{4g}-\frac{\pi r'^4\omega^2 w}{4g}\\&=\frac{\pi\omega^2 w\,(r^4-r'^4)}{4g}\\&=\frac{\pi\omega^2 w\,(r^2-r'^2)\,(r^2+r'^2)}{4g}.\end{aligned}$$

But $W=\pi\,(r^2-r'^2)\,w$; and if v and v' be the velocities of points on the outer and inner rims respectively, $v=r\omega$ and $v'=r'\omega$. Hence the work stored in the wheel is

$$\frac{W(v^2+v'^2)}{4g}.$$

257. When a body is moved against resistance by the agency of any machine, it will be found that the work done is always equal to the work applied. This is sometimes termed the PRINCIPLE OF WORK. This principle has already been verified in the case of simple machines whose parts are supposed to be without weight, and which work without friction. (Art. 111.) It may also be similarly verified for these machines when the weight of their parts is taken into account, remembering that when these are raised work is done, but that when they descend their latent energy is to be reckoned as part of the work applied.

The work expended in raising the parts of a machine, and in overcoming friction, is regarded as *lost work*, inasmuch as these are not the immediate purpose for which the machine is used. The work which is wholly expended in effecting the desired purpose is distinguished as *useful work*. The relation which subsists between the work applied to any machine, and the useful work accomplished by it, is termed the *modulus* of the machine.

258. *To find the relation between the applied and useful work when a body is drawn up a rough inclined plane by a power acting along the plane.*

Let W be the weight of the body, α the inclination of the plane, and μ the coefficient of friction.

Let P be the power necessary to move W up the plane, then (Art. 122) $P = W \sin \alpha + \mu W \cos \alpha.$

Let P move through the distance s, then $s . \sin \alpha$ will be the height through which W is raised. Then Ps is the work applied, and $Ws . \sin \alpha$ is the useful work. But

$$\begin{aligned} Ps &= Ws . \sin \alpha + \mu Ws . \cos \alpha. \\ &= Ws . \sin \alpha \, (1 + \mu \cot \alpha), \end{aligned}$$

or, $\qquad$ work applied = useful work × $(1 + \mu \cot \alpha)$.

The lost work, or $\mu Ws . \cos \alpha$, it may be noticed, is equal to the work which would be done in drawing the body along a horizontal plane of equal roughness through a distance $s \cos \alpha$, or a distance equal to the base of the plane, s being taken as the length of the plane.

259. In the preceding problem the applied and useful work are found to have a constant ratio to each other. It must not be assumed that this is the case. In general the relation between the applied and useful work will be exhibited in an equation of the form

$$u = \mathrm{A}u' + \mathrm{B}s,$$

where A and B are constants, u denoting the applied work, u' the useful work, and s the distance through which the power moves.

260. *To find the modulus of a lever of the first kind, the power and weight both acting vertically downwards.*

Let a and b be the arms of the lever, r the radius of the pivot, c the distance of the centre of gravity of the lever from the centre of the pivot, ϵ the angle of repose, and w the weight of the lever. Then (Art. 118)

$$\mathrm{P}(a - r\sin\epsilon) + w(c - r\sin\epsilon) = \mathrm{W}(b + r\sin\epsilon);$$

$$\therefore \qquad \mathrm{P} = \mathrm{W}\frac{b + r\sin\epsilon}{a - r\sin\epsilon} - w\,.\,\frac{c - r\sin\epsilon}{a - r\sin\epsilon}.$$

Let the lever move through an angle θ, then $a\sin\theta$ is the distance through which P moves, $b\sin\theta$ the distance through which W moves, and $c\sin\theta$ the distance through which w moves. But

$$\mathrm{P}\,.\,a\sin\theta = \mathrm{W}b\sin\theta\,.\,\frac{a(b + r\sin\epsilon)}{b(a - r\sin\epsilon)} - w\sin\theta\,.\,\frac{a(c - r\sin\epsilon)}{(a - r\sin\epsilon)}.$$

Adding $wc\sin\theta$ to both sides,

$$\mathrm{P}a\sin\theta + wc\sin\theta = \mathrm{W}b\sin\theta\,.\,\frac{a(b + r\sin\epsilon)}{b(a - r\sin\epsilon)} + wa\sin\theta\,.\,\frac{(a - c)r\sin\epsilon}{a(a - r\sin\epsilon)}.$$

But $\mathrm{P}a\sin\theta$ is the work done by P, and $wc\sin\theta$ is the work done by w, which in this case is to be considered as part of the applied work (Art. 257), and $\mathrm{W}b\sin\theta$ is the useful work. Hence the modulus required is

$$u = \frac{a(b + r\sin\epsilon)}{b(a - r\sin\epsilon)}u' + \frac{w(a - c)r\sin\epsilon}{a(a - r\sin\epsilon)}\,.\,s.$$

261. *To find the modulus of a wheel and axle when the power and weight both act vertically downwards.*

Let a be the radius of the wheel, b of the axle, r of the piv and let w be the weight of the wheel and axle. Then (Art. 11

$$P(a-r\sin\epsilon)=W(b+r\sin\epsilon)+wr\sin\epsilon;$$

or,
$$P=W\,.\,\frac{b+r\sin\epsilon}{a-r\sin\epsilon}+\frac{wr\sin\epsilon}{a-r\sin\epsilon}.$$

Let the wheel revolve through an angle θ, then $a\theta$ will be the distance through which P moves, and $b\theta$ that through which W moves, and consequently $Pa\theta=u$ and $Wb\theta=u'$. Multiplying both sides of preceding equation by $a\theta$ we have

$$Pa\theta=Wb\theta\,.\,\frac{a(b+r\sin\epsilon)}{b(a-r\sin\epsilon)}+\frac{wr\sin\epsilon}{a-r\sin\epsilon}a\theta;$$

or,
$$u=\frac{a(b+r\sin\epsilon)}{b(a-r\sin\epsilon)}u'+\frac{wr\sin\epsilon}{a-r\sin\epsilon}s.$$

In a similar manner it may be shown that when the power acts upwards, and W downwards,

$$u=\frac{a(b+r\sin\epsilon)}{b(a+r\sin\epsilon)}u'+\frac{wr\sin\epsilon}{a+r\sin\epsilon}s.$$

262. *The work done in raising a series of weights through various heights is equal to the work done in raising the sum of the weights through the height by which their common centre of gravity has been raised.*

Let W_1, W_2, &c., be the weights, and h_1, h_2, &c., the distances through which they are raised.

Let k_1, k_2, &c., be their heights over any horizontal plane when in their original position. Then, Art. 64, the height of their common centre of gravity is

$$\frac{W_1k_1+W_2k_2+\&c.}{W_1+W_2+\&c.}$$

In their new position the heights of the weights from the same plane are k_1+h_1, k_2+h_2, &c. Hence the height of the common centre of gravity in the new position is

$$\frac{W_1(k_1+h_1)+W_2(k_2+h_2)+\&c.}{W_1+W_2+\&c.}$$

nsequently if h be the height through which the common centre gravity has been raised is

$$h = \frac{W_1 h_1 + W_2 h_2 + \&c.}{W_1 + W_2 + \&c.}$$

$$\therefore \qquad (W_1 + W_1 + \&c.)h = W_1 h_1 + W_2 h_2 + \&c.$$

it $W_1 h_1$ is the work done in raising W_1, and $W_2 h_2$ the work done raising W_2, and so for the rest, and hence the proposition is oved.

263. Heat, as is seen in the steam-engine, has the power of ing mechanical work; and conversely, as we see in the case of iction, mechanical work may produce heat. It is therefore sirable to learn if possible the relation between heat and mechanil work. From experiments made by Mr. Joule it has been und that one unit of heat, *i.e.* the quantity of heat required to ise 1 lb. of water at its greatest density by 1° F., is equivalent 772 foot-pounds of work. Thus on the one hand the work pended in raising 772 lbs. through one foot will raise the temrature of 1 lb. of water by 1° F., and on the other hand the heat quired to raise the temperature of 1 lb. of water by 1° F. will ise 772 lbs. through one foot.

EXAMPLES.

1. If a horse moving at the rate of 5 miles per hour can do 848,000 units of work in half an hour, what force in pounds es the horse continuously exert? Ans. 140 lbs.

2. Find the work stored up in a cannon-ball of 161 lbs. moving the rate of 1 mile per minute. Ans. 19360 ft.-lbs.

3. A ball of 322 lbs. moving with a velocity of 100 feet per cond pierces an iron plate, and then moves on with a velocity of feet per second, what amount of work has been expended in rcing the plate? Ans. 45500 ft.-lbs.

4. A weight of 5 cwt. is drawn along a horizontal plane through 500 feet, the coefficient of friction is ·2; what is the work done?

Ans. 56000 ft.-lbs.

5. An engine draws a train weighing 40 tons along a level through 1000 feet, and in so doing imparts to the train a velocity of 12 miles per hour; what is the work done by the engine, the resistance from friction and other causes being 6 lbs. per ton?

Ans. 670970·4 ft.-lbs.

6. A uniform bar 10 feet in length, and weighing 24 lbs., revolves about its middle point at the rate of 6 revolutions per second; how many units of work are accumulated in it?

Ans. 4413·7

7. A fly-wheel of uniform thickness, and weighing 322 lbs., revolves at the rate of 8 revolutions per second, the radius of the outer rim is 10 feet, and of its inner rim 9 feet; how many units of work are stored in it? Ans. 1143295.

8. A fixed pulley whose radius is 6 inches, and weight 14 lbs., turns upon a pivot whose radius is 1 inch; how many units of work must be applied that a weight of 10 lbs. may be raised 39 feet, the coefficient of friction being ·15? Ans. 454.

9. Show that if a be the radius of a single moveable pulley, w its weight, r the radius of the pivot, and ϵ the angle of repose, the modulus, when the cords are parallel, is

$$u = \frac{a + r \sin \epsilon}{a} \,.\, u' + \frac{w(a + r \sin \epsilon)}{2a} \,.\, s.$$

10. If a weight is drawn up a rough inclined plane rising 1 in 240, and the resistance from friction is 7 lbs. per ton, show that the useful work is 4 sevenths of the applied work.

In working this and other examples, the student should remember that when an angle is small the cosine is nearly 1, and the sine is nearly equal to the tangent.

11. At what rate will an engine of 70-horse power draw a train weighing 60 tons up an incline of 1 in 320, the resistance from friction and other causes being 14 lbs. per ton?

Ans. $20\frac{5}{8}$ miles per hour.

12. A truck whose weight is 20 tons runs down an incline of 1 in 120; what amount of energy will it have acquired when it has run a distance of 1 mile, the resistance from friction, &c., being 16 lbs. per ton?

Ans. 281600 ft.-lbs.

13. A circular shaft 40 feet deep and 4 feet in diameter is half-full of water; how many units of work will be expended in emptying it?

Ans. 471238.5.

14. How much work is done in excavating a trench 20 feet long, 5 feet wide, and 8 feet deep, supposing that a cubic foot of the material weighs 120 lbs.?

Ans. 384000 ft.-lbs.

15. If $u = \mathrm{A}u' + \frac{1}{2}\mathrm{A}ws$ be the modulus of a single moveable pulley with its cords parallel, show that if three such pulleys be arranged as in the first system the modulus is

$$u = \mathrm{A}^3u' + \left(\frac{\mathrm{A}}{2} + \frac{\mathrm{A}^2}{4} + \frac{\mathrm{A}^3}{8}\right) ws.$$

HYDROSTATICS.

CHAPTER XVI.

ON THE FUNDAMENTAL PROPERTIES OF FLUIDS.

264. A fluid is a body all of whose parts can move freely amongst themselves. Motion consequently can be caused amongst the particles of a fluid body, by the application of the slightest conceivable force.

265. The science which treats of the equilibrium of forces acting upon fluid bodies is termed Hydrostatics.

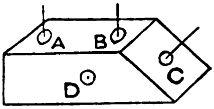
266. Fluids are divided into liquids and aëriform fluids. An aëriform fluid is distinguished from a liquid by the existence of an expansive or repulsive force amongst its particles, in consequence of which they tend in a greater or less degree to move off from one another.

267. When a fluid is at rest, any portion of it may be supposed to become a solid, without disturbing the equilibrium.

Since all the particles of the fluid are at rest, it is clear that they will not be less so if any number of them be rigidly connected with each other, and so deprived of the power of relative motion.

268. *When pressure is communicated to a fluid mass in equilibrium, it is transmitted equally and in all directions.*

n the sides of a closed vessel of any shape, let a number of rtures of equal area be made, and let se apertures be supplied with pistons ctly fitting them. Let the vessel be ed with water, and let the pistons be intained by some mechanical contri-



nce in their respective positions.* If any one of the pistons be essed in with a force P, it is found that all the other pistons perience a similar pressure, and that a corresponding force P ust be applied to each of them, in order to retain them in their aces. Thus the pressure communicated to the fluid has been nsmitted in all directions, for all the pistons experience it, atever their position, and whatever the shape of the vessel; and e transmitted pressure is equal to that communicated.

These results are the same, whatever the number of the pistons, d wherever placed. It follows, therefore, that every portion of e surface of the containing vessel, equal in area to that of the ston, experiences a pressure P. Thus, if the area of a piston is e square inch, and a pressure of 1 lb. is exerted upon it, a rresponding pressure of 1 lb. is experienced by every square inch the surface of the containing vessel; and similarly with any her pressure.

269. If A and B, the areas of whose lower surfaces are a and b pectively, be two pistons fitted into the upper surface of a vessel ed with fluid, and if Q be the pressure experienced by B, when ressure P is exerted upon A; then,

$$Q : P :: b : a.$$

For, in consequence of the pressure upon the piston A, a pressure is transmitted to every portion of the surface of the containing sel, whose area equals a, and therefore to every unit of area a ssure is transmitted equal to $P \div a$.

Since the lower surface of B contains b units of area, the whole pressure experienced by B is $Pb \div a$. But, by the hypothesis, Q denotes this pressure, therefore $Q = Pb \div a$; that is,

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* For the reason of this provision, see Art, 280.

$$Q : P :: b : a,$$

or,

$$\frac{\text{pressure on B}}{\text{pressure on A}} = \frac{\text{area of B}}{\text{area of A}}.$$

As an illustration, let the lower surface of the piston A be a circle whose diameter is 1 inch, and that of B a circle whose diameter is 1 foot, then

$$\frac{\text{area of B}}{\text{area of A}} = \frac{12^2}{1^2} = 144,$$

and, therefore,

$$\frac{\text{pressure on B}}{\text{pressure on A}} = 144,$$

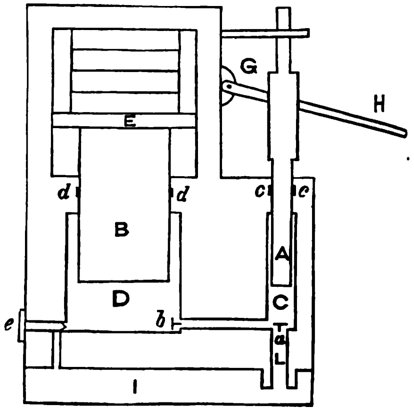
or,

$$\text{pressure on B} = 144 \times \text{pressure on A};$$

so that if a weight of 1 lb. rest upon the piston A, a weight of 144 lbs. must be placed upon the piston B in order to maintain equilibrium; and conversely, if 144 lbs. press on the piston B, the equilibrium may be maintained by a pressure of 1 lb. on A.

270. **Bramah Press.** This powerful machine, invented by Mr. Bramah, is an interesting application to practical purposes of the characteristic property of fluids.

C and D are two cylindrical vessels, connected with each other by means of the pipe *ab*. A and B are two solid pistons working in water-tight collars *cc*, *dd*. The piston B, the diameter of which is much greater than that of A, supports a plate E, upon which the substance to be pressed is placed. A is capable of being worked up and down by a lever GH, having its

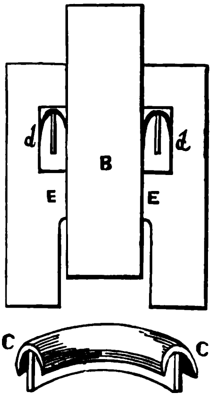


G. L is a pipe leading into a reservoir I, *a* is a valve
ward, and *b* a valve opening into the vessel D.
essel CD be supposed to be filled up with water, when
are in the position represented in the figure. Let A be
n with a pressure P, then, by the preceding Article, B
with a pressure equal to

$$P \times \frac{\text{area of B}}{\text{area of A}}.$$

ld to this pressure, the piston A will descend, and a
the water in C will be forced into the vessel D. The
he fluid will be prevented by the valve *b*. If the piston
aised, fresh water will be pumped up into the vessel C
servoir I, and if A be again forced down with a pressure
be again forced up with the same pressure as before.
s may be continued as long as the substance yields to
exerted upon it.
sure upon B may at any time be removed by unscrew-
g *e*, by which the water is allowed to flow back into the

consequence of the immense pressure exerted by the
ress, especial care is necessary to
escape of the water between the
the cylinder in which it works.
cted by the following contrivance.
piece of leather, after being well
formed into a collar, by doubling
rim of a metal cylindrical ring, in
r shown at *c c*, which represents
the collar and ring. The collar,
the ring, is placed in a recess cut
of the cylinder in which the pis-
as shown at *d d*. When the press
, the water forces its way through
E E, and enters the recess *d d*. The pressure of the



water, acting against the under surface of the leather, forces on one side of the ring against the piston, and on the other side against the cylinder, and by this means the escape of the water is effectually prevented. In fact, the greater the pressure, the more firmly is the leather pressed against the piston and the walls of the cylinder; so that, by this ingenious contrivance, the greater the danger of leakage, the greater is the protection provided against it.

272. The safety valve is another important application of the same principle. A portion of the boiler of a steam-engine, whose area contains a square inches, is furnished with a valve opening outwards. The valve is so contrived as to admit of being loaded with certain weights. If a weight of aP lbs. be placed on the valve, it is pressed down with a pressure of P lbs. to the square inch. If then at any time the pressure of the steam be greater than P lbs. to the square inch, the valve will be forced open, and the steam will escape; and since the pressure of the steam is the same on every inch of the surface of the containing vessel, no portion of the boiler will experience a greater pressure than P lbs. to the square inch. So that if P be less than the pressure per square inch which the weakest portion of the boiler can bear, the boiler can never burst.

EXAMPLES.

If two circular pistons, whose diameters are in the ratio of
are inserted in the sides of a closed vessel filled with a
tless fluid, what pressure will be necessary to keep the smaller
in its place when the larger piston is pressed in with a force
; lbs.? Ans. 45 lbs.

wo circular pistons are inserted in the sides of a closed
what must be the ratio of their diameters, in order that
he vessel is filled with any fluid, and a pressure of 9 oz. is
l to the smaller piston, a pressure of 64 lbs. may be trans-
to the larger? Ans. 3 : 32.

piston, whose area is 4 square inches, is inserted into one
a cubical vessel filled with water; required the pressure
he entire surface of the vessel when the piston is pressed in
force of 2 lbs., the edge of the vessel being 15 inches.
Ans. 6 cwt. 1 lb.

he diameter of the head of a cylindrical cask is 20 inches,
e height of the cask is 30 inches; what is the total pressure
he surface when a piston of one inch in diameter is pressed
a force P? Ans. 3199 P.

he diameter of the large piston in a Bramah press is 25
of the small piston ¼ inch, the lever is 2 feet 6 inches long,
attached to the small piston-rod at a point 2 inches from
crum; what weight would be raised by a power of 1 cwt.?
Ans. 7500 tons.

CHAPTER XVII.

ON THE EQUILIBRIUM OF FLUIDS ACTED UPON BY GRAVITY.

273. All the particles of a fluid, equally with those of any solid body, are subject to the action of gravity. Any portion, therefore, however small, of any fluid is drawn towards the earth with a certain degree of force, that is, is possessed of weight. Other forces, beside that of gravity, may act upon the particles of a fluid; and when that is the case, the determination of the conditions of equilibrium is difficult. We here confine ourselves to an examination of the circumstances which attend the equilibrium of fluids, when their particles are acted upon by the force of gravity alone.

274. *The surface of a fluid at rest is horizontal.*

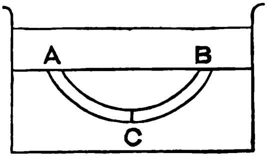
For, if possible, let any portion of the surface of a fluid at rest be not horizontal. There can then be taken in the surface three points which do not lie in the same horizontal plane. Let the fluid be supposed to be divided by a plane passing through these three points, and let the portion of fluid below and that above be supposed to become solid. The latter will then be a weight, unsupported by any force, at rest upon an inclined plane. But since there is no friction between the particles of a fluid, this is impossible. Therefore, no three points can be taken in the surface of a fluid at rest which do not lie in the same horizontal plane, and consequently the surface of a fluid at rest must be horizontal.

275. The total pressure of a fluid at rest, upon any surface in contact with it may be, and most commonly is, the sum of two different pressures; the one a pressure transmitted simply by the

d the other a pressure arising from the weight of the fluid The former, as we have seen, is the same upon every unit ace in contact with the fluid. The latter, it will be seen , varies with the position of the surface. When the known in any case, it is only necessary to add to it the of transmitted pressure corresponding to the area of the and the total pressure is found. If, then, we can determine sure of a fluid upon any surface arising from the action of upon its particles, upon the supposition that no external is communicated to the fluid, the problem is completely

The pressures of a fluid at rest at any two points in the rizontal plane are equal.

. and B be any two points in a fluid at rest, lying in the rizontal plane. Let P be the of the fluid upon a unit of A, and Q the pressure upon a area at B; then shall P and Q



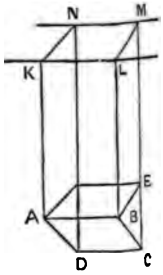
CB be any portion of the fluid from the rest and enclosed in ube, the parts AC, BC being perfectly symmetrical. Let lowest point of this tube, and let a vertical plate at C rigid. The pressures on each side of this plate arising fluid in ABC are equal, and consequently, since the fluid , the pressure transmitted from A and B must be equal, P and Q must be equal.

The pressure of a fluid at rest upon any plane must be in on perpendicular to the plane.

he only force opposed to the pressure of the fluid on the the resistance of the plane itself, and since there is equithese two forces must be equal and opposite. The resistance lane is perpendicular to the plane; this, consequently, must irection of the pressure of the fluid.

278. *To find the pressure of a fluid upon a rectangular plane immersed in it, so that one of its edges is parallel to the surface of the fluid.*

Let ABCD be the given plane, having AB parallel to the surface of the fluid. Let the vertical lines through A, B, C, D, meet the surface in K, L, M, N. Let the portion of the fluid included within the prism CK be supposed to become solid. The forces acting upon CK are the pressures on its surfaces and its own weight, and these are in equilibrium. The pressures on the vertical faces being (Art. 277) horizontal, produce no effect in the vertical direction. The only forces acting in the vertical direction are the vertical component of the pressure on ABCD, and the weight of the fluid mass CK; and, since there is equilibrium, these must be equal and opposite.



Let AE be a horizontal section of the prism through AB. Let $AB = a$, $BC = b$, and $BE = c$. Let h be the distance of the centre of ABCD from the surface of the fluid, P the pressure of the fluid upon ABCD, and w the weight of a cubic unit of the fluid. The content of CK equals the area of the face NL multiplied by its mean depth, or h;

$$\therefore \qquad \text{content of CK} = ach,$$

$$\therefore \qquad \text{weight of CK} = wach.$$

By Art. 98, the vertical component of P is $P \cdot \frac{BE}{BC}$, or $P\frac{c}{b}$;

$$\therefore \qquad P\frac{c}{b} = wach,$$

$$\therefore \qquad P = wabh.$$

But $wabh$ is the weight of a column of the fluid whose base is ab, or the plane ABCD, and whose height is h; and the centre of ABCD is also its centre of gravity. Therefore the pressure on the given plane is the weight of a column of the fluid whose base is the given plane, and whose height is the depth of the centre of gravity of the plane below the surface of the fluid.

If the vertical lines through A, B, C, D, do not meet the surface f the fluid mass, let LN be any horizontal plane, some portion of which lies vertically under the surface of the fluid mass, and let ' be the depth of this plane below the surface. Then, by what as been just proved, the pressure on any unit of area in this lane vertically under the surface of the fluid is wh'. Therefore, y Art. 276, the pressure on every unit of plane NL is wh', and onsequently the pressure on the area KLMN is $wach'$. Since this ressure acts vertically and concurrently with the weight of CN, the ertical component of P must be equal to the sum of these two forces;

$$\therefore \qquad P\frac{c}{b} = wach + wach',$$

$$\therefore \qquad P = wab\,(h + h');$$

nd $h + h'$ is the entire depth of the centre of gravity of ABCD elow the surface of the fluid. In like manner it may be shown, hat in all cases the pressure of a fluid on any rectangular plane, aving one of its sides parallel to the surface of the fluid, is the eight of a column of the fluid whose base equals the area of the lane, and whose height is the depth of the centre of gravity of he plane below the surface of the fluid.

279. *To determine the total pressure of a fluid upon any surface.* Let the surface be divided into small rectangular planes, each aving one of its sides parallel to the surface of the fluid. Let A e the area of the whole plane, and z the distance of its centre of ravity below the surface of the fluid. Let A_1, A_2, A_3, &c., be the reas of the several rectangles, and z_1, z_2, z_3, &c., the distances of eir centres of gravity below the surface of the fluid; then,

$$\text{pressure on given surface} = wA_1z_1 + wA_2z_2 + wA_3z_3 + \&c.$$
$$= w(A_1z_1 + A_2z_2 + A_3z_3 + \&c.)$$

But (Art. 64) $\qquad Az = A_1z_1 + A_2z_2 + A_3z_3 + \&c.;$

$$\cdot \qquad \text{pressure on given surface} = wAz,$$

r the pressure on the surface is the weight of a column of the uid whose base equals the area of the given surface, and whose eight is equal to the depth of the centre of gravity of the surface elow the surface of the fluid.

Ex. 1. To find the pressure on a rectangular plane 10 in. by 4, when immersed in water so that its centre of gravity is at a depth of 20 inches.

The area of the plane is 10 × 4, or 40 square inches. Hence the pressure on the plane is the weight of 40 × 20, or 800 cubic inches of water.

The weight of a cubic foot of pure water is 1000 oz. avoirdupois, and therefore the weight of a cubic inch is 1000 ÷ 1728 oz.

$$\therefore \quad \text{pressure required} = \frac{800 \times 1000}{1728}$$
$$= 462 \cdot 96 \text{ oz.}$$

Ex. 2. To compare the pressures on the bottom and side of a cubical vessel filled with any fluid.

Let a be the length of the edge of the cube, and w the weight of a cubic unit of the fluid. The area of the bottom of the vessel is a^2, and the depth of its centre of gravity is a, therefore the pressure on the bottom is equal to wa^3.

The area of the side of the vessel is a^2, and the depth of its centre of gravity is $\frac{1}{2}a$, therefore the pressure on the side is equal to $\frac{1}{2}wa^3$. Hence the pressure on the side of the vessel is one-half of that on the bottom.

280. If the student will now revert to Art. 268, he will see why it was necessary to insert the provision, that the pistons be maintained in their position by some mechanical contrivance. For if the pistons (as in the case with the pistons C, D) be situated anywhere except upon the uppermost surface of the vessel, they will experience a pressure greater or less according to their distance below the surface of the fluid, and will, in consequence of this pressure, be forced out, if not prevented.

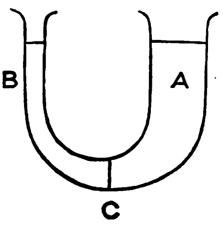
The pistons A and B, which press upon the uppermost surface of the fluid, experience no pressure from the weight of the fluid itself; whatever force, therefore, is impressed upon them is transmitted undiminished to every part of the fluid. But if a pressure applied elsewhere, at C for instance, and no other force act upon e piston, then a part of this pressure will be employed in counter-alancing the pressure of the fluid upon C, and the remainder only ill be the pressure transmitted by the fluid. Hence, if a be the ea of the piston, h the distance of its centre of gravity below the urface of the fluid, and w the weight of a cubic unit of the fluid, d if aP be the pressure exerted upon the piston, the pressure ansmitted to every corresponding area is $a\mathrm{P} - wah$, or the pressure ansmitted to every unit of area is $\mathrm{P} - wh$.

281. Since the pressure of a given fluid upon a given horizontal ane depends only upon the depth of the plane beneath the surce of the fluid, it follows that, if the bottoms of any number of ssels of various shapes be of equal area, and the same fluid be ured to the same height in all, the pressure upon the bottom of ch vessel will be the same, however different the quantity of uid in each may be. If the bottoms of the vessels be moveable, will be found by experiment that the same force is necessary in ch case to retain them in their respective positions.

282. *If a fluid be poured into any one of a number of open ssels having a free communication with each other, the fluid will e to the same height in all.*

Let A and B be any two such vessels, having the lowest point C.

Let a thin vertical plate of the fluid passing through C become gid. Let a be the area of this plate, h the distance of its centre gravity below the surface of the fluid in , and h' its distance below the surface B.



Let w be the weight of a cubic unit of e fluid. Then the pressure exerted on e one side of the plate by the fluid in A wah, and that exerted on the other side y the fluid in B is wah'. But since there is equilibrium, these pressures must be equal; therefore

$$wah = wah',$$

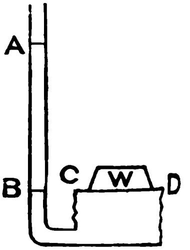
or,

$$h = h'.$$

283. Hydrostatic Bellows. AB is a narrow tube communicating with a vessel formed by uniting together two pieces of wood by some flexible and waterproof subtance. If water be poured down the tube, it will enter the vessel, and raise a large weight W placed upon CD.

When equilibrium exists, let A be the surface of the water in the tube, and let B lie in the same horizontal plane with CD. Let w be the weight of a cubic unit of water, and let h be the depth of B or CD below A.

The pressure upon each unit of area in CD is wh. Therefore, if b = number of units in the area of CD, the total pressure on CD = wbh; and, since there is equilibrium, this must equal the weight supported; therefore



$$W = wbh.$$

If a = area of the horizontal section of the pipe at B, weight of fluid in AB = wah;

$$\therefore \quad W = \frac{b}{a} \times \text{weight of fluid in AB}.$$

If CD and the tube are circular, and if R be the radius of CD, and r the radius of horizontal section of the tube,

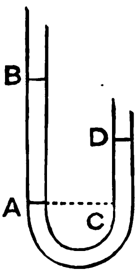
$$b : a :: R^2 : r^2,$$

$$\therefore \quad W = \left(\frac{R}{r}\right)^2 \times \text{weight of fluid in AB}.$$

284. *To determine the position of equilibrium when two fluids which do not mix meet in a bent tube.*

Let A be the common surface to the two fluids. Let B be the surface of the one fluid, and D that of the other, when there is equilibrium.

Let h be the distance of B above the horizontal plane at A, and h' the distance of D above the same plane.



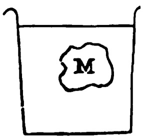
Let w be the weight of a cubic unit of the fluid in AB, and w' of the fluid in ACD. Let a be the area of the horizontal section of the tube at A. If a thin plate of the fluid at A be supposed to become solid, since there is equilibrium, the pressures on its upper and under surfaces must be equal. The pressure on its upper surface is wah, and on its lower surface $w'ah'$; therefore

$$wah = w'ah',$$
$$wh = w'h';$$
$$\therefore \quad \frac{h}{h'} = \frac{w'}{w}.$$

285. The pressure of a fluid at any point of a surface acts in the direction of the normal to the surface at that point. The pressure found in Art. 279 is the *total pressure*, or the sum of the pressures at all points of the surface. The resultant of any number of forces is equal to their sum only when the forces are parallel, and hence, the total pressure already found will be the *resultant pressure* only when all the normals are parallel, that s, when the surface is a plane. If the surface be not a plane, the resultant pressure will not be the same as the total pressure, but ess than it, since the resultant of any number of forces which are nclined to each other is less than their sum. The determination of the resultant pressure requires in general a knowledge of the differential calculus. There are, however, certain cases in which the resultant pressure may be more easily found. These may be classified under the following two divisions: first, those in which the given surface is the entire surface in contact with the fluid, when the body is either wholly or partially immersed in the fluid; and secondly, those in which the given surface is bounded by one or more *plane* curves.

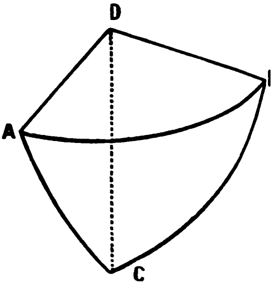
286. *To find the resultant pressure of a fluid on the surface of a solid, wholly or partially immersed in it.*

Let any portion M of any fluid at rest become solid, and sin the equilibrium is not thereby disturbed, the forces acting upon M must be in equilibrium. The only forces acting upon M are the weight of M acting vertically downwards at the centre of gravity of M, and the pressures of the fluid upon the surface of M. Hence, the resultant of these pressure must be equal and opposite to the weight of M acting at its centr of gravity. But the fluid will exert the same pressure upon th surface of any other body of the same form as M occupying i place. Hence, the resultant of the pressure of a fluid on th surface of a solid immersed in it is equal to the weight of the flui displaced, and acts vertically upwards through the centre of gravi of the fluid displaced.



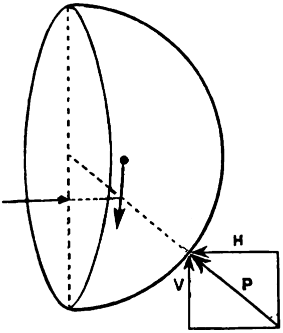
287. When the given surface is not the entire surface in conta with the fluid, the resultant pressure may be found by the follo ing method, if the surface be bounded by plane curves.

Let ABC be the given surface, bounded by the plane curves A BC, and CA. Let planes passing through these curves intersect in AD, DB, and DC, and let ADBC be a portion of the fluid mass, bounded by these planes and the given surface. Let this portion of the fluid be supposed to become solid. It is then a body at rest under the action of the following forces:—the weight of the fluid mass, the pressures on the planes, and the pressure on the give surface. The last, therefore, must be equal and opposite to th resultant of the rest. By finding then the weight of the flui mass, and compounding it with the pressures on the plane surface we obtain the required resultant pressure on the given surface.



In applying this method, it will often be convenient to find th required pressure by finding separately its horizontal and vertical components; the former being equal to the resultant of the other horizontal pressures acting on ABCD, and the latter equal to the resultant of the other vertical pressures.

Ex. 1. A sphere is immersed to a given depth in a given fluid, to find the magnitude and direction of the resultant pressure on a hemisphere whose bounding plane is vertical.



Let r = radius of sphere, h = depth of the centre, and w = the weight of a unit of the fluid, then the pressure on the bounding plane is equal to

$$w\pi r^2 h,$$

and acts horizontally; and the weight of the fluid included between the hemisphere and its bounding plane is equal to

$$\tfrac{2}{3}w\pi r^3,$$

and acts vertically. These two forces are in equilibrium with the required pressure (P). Therefore, if V be the vertical and H the horizontal components of P,

$$V = \tfrac{2}{3}w\pi r^3,$$

and

$$H = w\pi r^2 h.$$

Therefore,

$$P = \sqrt{(V^2 + H^2)},$$
$$= \tfrac{1}{3}w\pi r^2\sqrt{(4r^2 + 9h^2)};$$

and if θ be the inclination of P to the vertical line,

$$\tan\theta = \frac{H}{V} = \frac{3h}{2r}.$$

COR. If the sphere be just immersed, then $h = r$, and resultant pressure $= \tfrac{1}{3}w\pi r^3\sqrt{13}$.

Ex. 2. A sphere is immersed to a given depth in a fluid, to find the resultant on either of the four parts into which the surface is divided by two vertical planes passing through the centre, and at right angles to each other.

Let r = radius of sphere, h the depth of the centre, and weight of a unit of the fluid.

The pressure on each of the bounding planes is equal to

$$\tfrac{1}{2}w\pi r^2 h.$$

These pressures are both horizontal, and at right angles to other, therefore the resultant horizontal pressure is

$$\tfrac{1}{2}\sqrt{2}\,.\,w\pi r^2 h.$$

The weight of the fluid included between the given surfac the bounding planes is $\tfrac{1}{3}w\pi r^3$.

Hence, as before, $V = \tfrac{1}{3}w\pi r^3$,

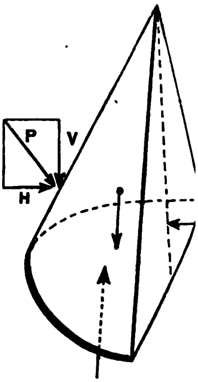
$H = \tfrac{1}{2}\sqrt{2}\,.\,w\pi r^2 h$,

therefore, $P = \tfrac{1}{6}w\pi r^2\sqrt{(4r^2 + 18h^2)}$.

Cor. If the sphere be just immersed, then

$$\text{resultant pressure} = \tfrac{1}{6}w\pi r^3\sqrt{22}.$$

Ex. 3. A cone immersed in a fluid, with its vertex on the su and its axis vertical, is bisected by a plane passing through the axis, to find the resultant pressure on either half of the conical surface.



Let h = height of the cone, r = radius of base, and w the weight of a unit of the fluid. Then the pressure on the vertical plane is equal to

$$\tfrac{2}{3}wrh^2,$$

and acts horizontally. The pressure on the base is $\tfrac{1}{2}w\pi r^2 h$, and acts vertically upwards, and the weight of the included fluid is

$$\tfrac{1}{6}w\pi r^2 h.$$

Hence, $V = \tfrac{1}{2}w\pi r^2 h - \tfrac{1}{6}w\pi r^2 h$,

$= \tfrac{1}{3}w\pi r^2 h$,

and $H = \tfrac{2}{3}wrh^2$.

$\therefore$ $P = \tfrac{1}{3}wrh\sqrt{(4h^2 + \pi^2 r^2)}$.

288. If any body hang freely by a cord, we have seen (Art. 55) that the centre of gravity lies in the vertical line drawn through the point of suspension, that is, in the line of the cord; and the tension in the cord equals the weight of the body.

If such a body be wholly immersed in a fluid, it has been shown in Article 286, that it will be pressed upwards by a force equal to the weight of the fluid displaced, which in this case is the weight of a portion of the fluid equal to the bulk of the body; and this pressure is directly opposite to the weight of the body; therefore the tension in the cord will be the weight of the body diminished by the weight of an equal bulk of the fluid.

289. *To determine the conditions of equilibrium of a floating body.*

If a body floating in a fluid be at rest, the forces acting upon it are the weight of the body, which acts as its centre of gravity, and the resultant pressure of the fluid which acts at the centre of gravity of the fluid displaced.

The necessary conditions of equilibrium are, that these forces be *qual* and *opposite*. The two conditions therefore are,

First, That the weight of the body be equal to that of the fluid isplaced; and,

Secondly, That the centres of gravity of the body and the fluid isplaced be in the same vertical line.

As a body cannot displace a quantity of fluid greater than its wn bulk, if the weight of the body be greater than that of an ual bulk of the fluid, the body cannot float in that fluid. If the eight of the body be exactly equal to that of an equal bulk of e fluid, the body will have in no position any tendency either to e or to sink, provided only it be entirely covered by the fluid.

EXAMPLES.

1. To what depth must a surface be sunk in water that *the* pressure upon it may be at the rate of 25 lbs. to the square inch?

Ans. $57\frac{3}{5}$ feet.

2. If the side of a vessel be a triangle, having its base at the bottom of the vessel, show that the pressure exerted upon it by any fluid when the vessel is full is to that exerted when the vessel is filled to only half its depth, as 16 : 5.

3. If in the preceding the base of the triangle be the top of the vessel, show that the pressure when the vessel is full is to the pressure when the vessel is only half filled, as 8 : 1.

4. Show generally, in the last example, that the pressure on the side when the vessel is full is to the pressure when the vessel is filled to one *n*th of its depth, as $n^3 : 1$.

5. Find the total pressure upon the surface of a globe, 2 feet in diameter, when immersed in water so as to be just covered.

Ans. 785·39 lbs.

6. A globe, 2 feet in diameter, when floating in water, is half immersed, what is its weight? Ans. 130·9 lbs.

7. A conical body floats in water with its vertex downwards, what will be the depth of the vertex below the surface of the fluid, the weight of the cone being 200 oz., its height 24 inches, and the diameter of its base 12 inches? Ans. $\sqrt[3]{(3{\cdot}05)}$ ft.

8. A cylindrical vessel is filled with water, what is the ratio of the pressure upon the bottom to that upon the side; r being the radius of the base, and h the height of the vessel.

Ans. As $r : h$.

9. A cylindrical vessel is filled half with water and half with a fluid twice as heavy as water; the two fluids do not mix; compare the pressures upon the upper and lower half of the sides.

Ans. The pressure upon the lower half is 4 times that upon the upper half.

10. A sphere, whose radius is 6 inches and weight 35 lbs., is suspended by a string; required the tension in the string when he sphere is wholly immersed in water. Ans. 2 lbs. 4·4 oz.

11. A rectangular plate, ABCD, hangs by a cord at A with its lane vertical, and is partly immersed in water, in the position of quilibrium the diagonal BD coincides with the surface of the ater; required the weight of the plate and the tension in the ring.

Let *a* and *b* be the sides of the plate in inches, and *c* its thickss; then, if *w* be the weight of a cubic inch of water, the weight the plate is equal to $\frac{2}{3}$ *wabc*, and the tension in the string is ual to $\frac{1}{6}$ *wabc*.

12. If a triangular plate ABC, having its sides AB, AC unequal, suspended at A, and be partially immersed in any fluid, show at in the position of equilibrium the line drawn from A to the section of the opposite side cannot be vertical.

13. If any plate be suspended, and partially immersed in any id then in the position of equilibrium, the moment of the volume the displaced fluid about the point of suspension is to the moment the volume of the plate as the weight of the plate is to the eight of an equal volume of the fluid.

Let V and V′ be the volumes of the entire plate, and of the mmersed part, and q, q', the distances of their centres of gravity om the vertical line through the point of suspension. Let W be e weight of the plate, and w the weight of a unit of the fluid; en the upward thrust of the fluid is equal to

$$wV'.$$

The resultant of the weight of the plate and the thr fluid must pass through the point of suspension; therefo the moments about this point, we have

$$wV'q' = Wq.$$

Whence,
$$\frac{V'q'}{Vq} = \frac{W}{wV}.$$

14. In the preceding, if V′, q', apply not to the part but to the part not immersed, show that

$$\frac{V'q'}{Vq} = 1 - \frac{W}{wV}.$$

15. A triangular plate ABC, having the angle at B a ri is suspended at A, and partially immersed in a fluid; re height of A above the surface of the fluid, in order th position of equilibrium the side AB may be vertical.

Let $AB = a$; let W be the weight of the plate, an weight of an equal volume of the fluid; the required d equal to
$$a\sqrt[3]{\left(1 - \frac{W}{W'}\right)}.$$

16. A conical vessel having its vertex downwards, with a given fluid, is bisected by a vertical plane passin the axis; to find the resultant pressure on either porti surface.

Let h be the height of the cone, r the radius of its b the weight of a unit of the fluid, then the required p equal to
$$\tfrac{1}{6}wrh\sqrt{(4h^2 + \pi^2r^2)}.$$

17. In the preceding, if the cone be divided into f portions by two planes through the axes at right angle other, show that the resultant pressure on each portic conical surface is equal to

$$\tfrac{1}{12}wrh\sqrt{(8h^2 + \pi^2r^2)}.$$

CHAPTER XVIII.

ON SPECIFIC GRAVITY.

290. The specific gravity of any substance is the ratio of the weights of equal volumes of that substance, and of a certain standard substance.

The standard for solids and liquids is distilled water, at a temperature of 60°.

The standard for gases is pure atmospheric air, at a temperature of 60°, with the barometer at 30 inches.

Hence, if the specific gravity of any solid or liquid be 2, the weight of a cubic inch of the substance is twice the weight of a cubic inch of water. Or generally, if s be the specific gravity of any substance, and w the weight of a cubic unit of the standard, the weight of a corresponding unit of the substance is sw; and if V be the volume of the substance, and W its weight,

$$W = Vsw.$$

A cubic foot of pure water, at a temperature of 60°, weighs 1000 ounces avoirdupois; and 100 cubic inches of air, at the standard temperature and pressure, weighs 31 grains.*

The absolute weight of any substance may hence be easily found from a table of specific gravities.

For example, to find the weight of a cubical block of marble, whose side is 4 feet, the specific gravity of the marble being 2·7.

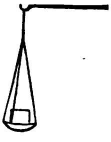
The block contains 4^3, or 64 cubic feet of marble. Therefore,

$$\begin{aligned} \text{weight of the block} &= 64 \times 2{\cdot}7 \times 1000 \text{ oz.} \\ &= 10800 \text{ lbs.} \\ &= 4 \text{ tons } 16 \text{ cwt. } 1 \text{ qr. } 20 \text{ lbs.} \end{aligned}$$

---

* More accurately, 31·0117 grains. It will be convenient to remember that a cubic foot of air weighs very nearly one-thirteenth of a pound. This is slightly in excess; when greater accuracy is required, subtract from the result obtained by this rule its 200th part.

291. The hydrostatic balance is, in its simplest form, pair of scales, with a fine thread attached to the under surface of one of the scale-pans. By this arrangement, any solid substance can be weighed, both in the ordinary way, and also by means of the thread when immersed in any liquid.



The loss of weight when a body is weighed in any liquid is equal to the weight of the liquid displaced. Hence, if V be the the body, and s' the specific gravity of the liquid,

$$\text{loss of weight} = Vs'w.$$

292. *To find the specific gravity of a body heavier tha*

Weigh the body both in air and in water. Let W be i and l the loss of weight in water. Then (Art. 291) $l =$ s be the specific gravity,

$$s = \frac{W}{Vw} = \frac{W}{l}.$$

If the body be weighed, not in water, but in some ot whose specific gravity is s', then, if l' be the loss of weight, and therefore

$$s = \frac{W}{l'} \times s'.$$

293. *To find the specific gravity of a body lighter tha*

To the given body attach some other body heavy enou it in water; weigh the two together both in air and in w loss of weight is equal to the weight of the water dis both. Then weigh the heavy body in air and in water of weight is equal to the weight of the water displace heavy body.

The difference of the two losses is therefore equal to t of water displaced by the given body. Hence,

$$\text{sp. gr.} = \frac{\text{weight of body}}{\text{difference of the two losses}}.$$

294. *To find a relation between the weights and specific gravities of the components, and the weight and specific gravity of the compound.*

Let W_1, W_2, &c., be the weights; V_1, V_2, &c., the volumes; and s_1, s_2, &c., the specific gravities of the components. Let W be the weight, V the volume, and s the specific gravity of the compound.

Then if no change of volume result from the composition,

$$V = V_1 + V_2 + \&c.$$

But by Art. 290, $V = \frac{W}{sw}$; $V_1 = \frac{W_1}{s_1 w}$, and so on. Therefore

$$\frac{W}{sw} = \frac{W_1}{s_1 w} + \frac{W_2}{s_2 w} + \&c.$$

$$\therefore \quad \frac{W}{s} = \frac{W_1}{s_1} + \frac{W_2}{s_2} + \&c.$$

295. If the volume be changed by the composition:—let the volume after composition be to the sum of the volumes of the components in the ratio of $m : n$. Then

$$V = \frac{m}{n}(V_1 + V_2 + \&c.)$$

$$\therefore \quad \frac{W}{s} = \frac{m}{n}\left(\frac{W_1}{s_1} + \frac{W_2}{s_2} + \&c.\right)$$

296. *To find a relation between the volumes and specific gravities of the components, and the volume and specific gravity of the compound.*

Using the same notation as in Art. 294,

$$W = W_1 + W_2 + \&c.$$

But, Art. 290, $W = Vsw$; $W_1 = V_1 s_1 w$, and so on. Therefore

$$Vsw = V_1 s_1 w + V_2 s_2 w + \&c.$$

$$\therefore \quad Vs = V_1 s_1 + V_2 s_2 + \&c.$$

297. *To determine the specific gravity of a liquid by means of the specific gravity bottle.*

The specific gravity bottle is simply a small flask fitted with a ground stopper.

Let w be the weight of the flask, x its weight when filled with the given liquid, and y its weight when filled with distilled wate Then $x-w$ is the weight of the given liquid contained in the flas and $y-w$ the weight of a similar volume of distilled water.

$$\therefore \qquad \text{sp. gr. of the liquid} = \frac{x-w}{y-w}.$$

Specific gravity bottles are very frequently made so as to contain exactly 1000 grains of pure distilled water. Then, if x and w are known in grains,

$$\text{sp. gr. of the liquid} = \frac{x-w}{1000}.$$

298. *To determine the specific gravity of a liquid by weighing a solid body in it.*

Let V be the volume of the body, l its loss of weight when weighed in water, and l' its loss of weight when weighed in the given liquid. Then, if s' be the specific gravity of the liquid,

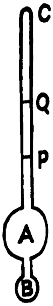
$$Vs'w = l'.$$

And
$$Vw = l.$$

$$\therefore \qquad s' = \frac{l'}{l}.$$

299. *To determine the specific gravity of a liquid by means of the common hydrometer.*

The common hydrometer consists of a small hollow sphere A, to which there is attached on one side a slender graduated stem, and on the other a smaller sphere B, made of such a weight as to allow the whole instrument to float with the sphere A entirely immersed.



A table accompanies the instrument, in which are given the specific gravities corresponding to the several divisions of the stem. The specific gravity of any liquid is found by observing the degree to which the instrument sinks when placed in it, and then referring to the tables.

To explain the construction of the tables; let the instrument sink to P when placed in distilled water, and to Q when in some other liquid whose specific gravity, say, is s. Let V be the volume of the hydrometer, and a the area of the section of the stem. Let $CP = x$, and $CQ = y$, and let w be the weight of a unit of water.

By Art. 289, when a body floats, the weight of the body is equal to that of the fluid displaced.

The quantity of water displaced is equal to the volume of the hydrometer, minus the part CP, that is, is equal to $V - ax$. Therefore the weight of the displaced water is equal to

$$(V - ax)w.$$

Similarly the volume of the fluid displaced is $V - ay$, and therefore the weight of the fluid displaced is equal to

$$(V - ay)sw.$$

But the weights of the displaced water and the displaced fluid are equal, since each is equal to the weight of the instrument. Therefore,

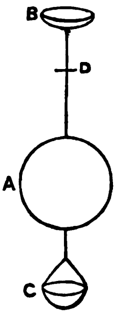
$$(V - ay)sw = (V - ax)w.$$

$$\therefore \quad s = \frac{V - ax}{V - ay}.$$

The tables are then formed by calculating the value of this fraction for different values of y.

300. *To determine the specific gravity of a fluid by means of Nicholson's hydrometer.*

Nicholson's hydrometer consists of a hollow sphere, or some other symmetrical figure, connected to a stem which carries a cup at each end. The lower cup is loaded, so as to secure stability of equilibrium when the instrument floats with the stem vertical.



When in use, the instrument is sunk to an invariable depth, by means of a weight placed in the upper cup; the volume of fluid displaced is therefore always the same.

Let W be the weight of the instrument, and w the weight which must be placed in B to sink the instrument to the point D in distilled water; then the weight of water displaced is equal to

$$W + w.$$

Let w' be the weight required to sink the instrument to D when placed in any given fluid, then the weight of the fluid displaced is equal to $W + w'$.

The volumes displaced in each case are the same, therefore

$$\text{sp. gr. of the fluid} = \frac{W + w'}{W + w}.$$

301. *To find the specific gravity of a solid by means of Nicholson's hydrometer.*

Let W be the weight of the instrument, and w the weight required to sink it to D in the standard fluid; then the weight of water displaced by the instrument is

$$W + w.$$

Let the given solid be placed in the upper cup, and let w_1 be the weight which sinks the instrument to D; then the weight of the solid is equal to $w - w_1$.

Let the given solid be placed in the lower cup, and let w_2 be the weight which sinks the instrument to D; then the weight of the water displaced by the instrument and the solid together is equal to $W + w - w_1 + w_2$.

Hence, the weight of water displaced by the solid alone is equal to

$$w_2 - w_1.$$

Consequently, the specific gravity of the solid is equal to

$$\frac{w - w_1}{w_2 - w_1}.$$

302. The method of finding the specific gravity of a gas is similar to that of finding the specific gravity of a liquid by means of the specific gravity bottle. Instead of the bottle a flask is used, fitted with a stop-cock, and capable of being screwed on to the end of an exhausting syringe, or to the plate of an air-pump.

The air is first exhausted from the flask, and the stop-cock being closed, the flask is carefully weighed. We thus obtain the weight of the empty flask. The flask is then weighed when filled with air and with the gas. Deducting from these weights the weight of the

k, we obtain the weight of the air and of the gas con-
he flask; and the volumes being equal,

$$\text{sp. gr.} = \frac{\text{weight of gas in the flask}}{\text{weight of air in the flask}}.$$

volume of the flask will commonly differ greatly from
e weight by which it is balanced, the flask and the
ll be differently affected by the buoyancy of the sur-
air; and although this will not sensibly affect the
f the result if the atmospheric pressure remain the
ughout the entire series of
it is otherwise if any change
ospheric pressure takes place.
en great accuracy is required,
must be made to guard against
of error. This is conveniently
ttaching to the scale-pan (*b*)
the weight, a closed flask
exterior volume is equal to
he flask (A) containing the
arms of the balance are thus
ffected by the atmospheric
nder all circumstances, and the difference between the
(*b*) when the flask (A) is exhausted, and the weight
flask is filled with any gas, will give the weight of the
gas. It is almost needless to state that in such experi-
ry sensitive balance should be used.

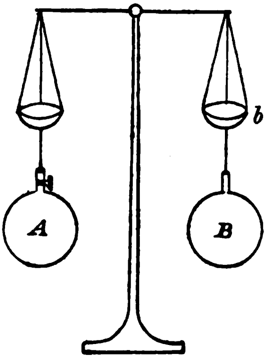


TABLE OF SPECIFIC GRAVITIES.

Platinum	21·47	Porcelain 2·15 to	2·38
Gold	19·30	Brick	2·00
Mercury (congealed)	15·61	Ivory	1·82
,, (fluent)	13·60	Phosphorus	1·77
Lead	11·38	Sugar	1·61
Silver	10·50	Lignum Vitæ	1·33
Bismuth	9·88	Box (Dutch)	1·33
Copper	8·67	Ebony (Indian)	1·21
Brass 7·82 to	8·40	Mahogany ·64 to	1·06
Steel	7·83	Blood (human)	1·05
Iron (forged)	7·79	Milk	1·03
Tin	7·29	Sea Water	1·03
Iron (cast)	7·25	Sodium	·93
Zinc (compressed)	7·19	Proof Spirit	·92
,, (common)	6·90	Box (French)	·91
Fluor Spar 3·09 to	3·79	Potassium	·87
Quartz 2·62 to	3·75	Beech	·85
Porphyry 2·5 to	3·00	Alcohol	·80
Granite 2·61 to	2·96	Plum Tree	·78
Plate Glass	2·94	Maple	·75
Rock (crystal) 2·58 to	2·89	Scotch Fir	·69
Marble	2·71	Cedar	·60
Green Glass	2·64	Spruce Fir	·52
Portland Stone	2·50	Cork	·24

EXAMPLES.

1. What is the weight of a block of gold 12 inches long, 8 wide, and 6 deep?
Ans. 3 cwt. 66 lbs. 1¾ oz.

2. What is the weight of a silver globe, whose radius is 3 inches?
Ans. 687·22 oz. avoir.

3. A block of granite weighs 7½ cwt.; determine its volume in cubic feet, the specific gravity of the granite being 2·8.
Ans. 4·8 cubic feet.

4. A body weighs in water 3 oz. less than in air, what is its volume?
Ans. 5·184 cubic inches.

onical piece of cork, whose height is 12 inches, and specific 16, floats in water with its vertex downwards; what will pth of the vertex below the surface of the fluid?

Ans. 7·2 inches.

conical body, whose height is *h*, and specific gravity *s*, its vertex downwards in a liquid whose specific gravity that the depth of the vertex below the surface is

$$h \, . \sqrt[3]{\left(\frac{s}{s'}\right)}.$$

ne, 10 inches in height, when floating in water with its wards, is immersed to the depth of 6 inches; to what l it be immersed when floating in a fluid whose specific ·943? Ans. 8 inches.

6 oz. of copper lose 3 oz. in water, and 7 oz. of zinc lose d if an alloy of copper and zinc weighing 80 oz. lose at is the proportion of copper in the alloy?

Ans. 13 : 7.

substances, A and B, whose specific gravities are s_1 and each other *in water*; determine the ratio of the weights B. Ans. $s_1(s_2 - 1) : s_2(s_1 - 1)$.

e specific gravity of some adulterated milk is found to be hat proportion of water does it contain?

Ans. One pint of water in every 5 of the mixture.

termine the greatest weight of Portland stone which can by 100 cubic inches of cork. Ans. 4 lbs. 9·3 oz.

rown composed of gold and silver, and weighing 4 lbs. is, displaces, when entirely immersed, 7 cubic inches, a old of the same weight displaces $5\frac{3}{4}$ inches, and a lump f the same weight displaces $10\frac{1}{2}$ inches; determine the gold contained in the crown. Ans. 2 lbs. $15\frac{3}{19}$ oz. av.

13. A Nicholson's hydrometer, weighing 200 grains, requi weight of 66 grains to sink it to the standard depth in water, a weight of 44 grains to sink it to the same depth in spirit wine; determine the specific gravity of the spirits of wine?

Ans. ·917.

14. The specific gravity of a mixture of a pint of sulphuric ac and a pint of water is 1·71, the sp. gr. of the sulphuric acid bein 1·85; determine the loss of volume resulting from the mixture.

Ans. $\frac{1}{3}$ pint, or 11·52 cubic inches.

15. A body weighing 450 grains loses 210 grains in water, an 150 grains in spirit; find the sp. gr. and volume of the body, an the sp. gr. of the spirit. Ans. 2·143; ·829 cubic inches; ·714.

16. A substance placed in the upper cup of a Nicholson's hydr meter requires 5 grains to sink it to the standard depth in a flui whose specific gravity is ·918, and 50 grains when placed in th lower cup; determine the specific gravity of the substance, th weight of the hydrometer being 240 grains, and the weight requir to sink it to the standard depth in the given fluid 30 grains.

Ans. ·51.

17. If s be the specific gravity of an alloy compounded of tw metals whose specific gravities are s_1 and s_2 respectively, what is th proportion of the two metals? Ans. As $s_1(s-s_2) : s_2(s_1-s)$.

18. How much cork (sp. gr. ·24) must be attached to 114 oz. o a metal whose specific gravity is 7·2 that the two together ma just float in water? Ans. 31 oz.

CHAPTER XIX.

ON ATMOSPHERIC PRESSURE.

. The atmosphere surrounding the earth, being a fluid acted gravity, presses upon all bodies immersed in it in accordance he general laws of fluid pressure.

pressure of the atmosphere upon any body will consequently ith the depth of that body below its highest surface. As, r, the dimensions of any ordinary body upon the surface of th are inconsiderable, when compared with the height of nosphere, this pressure may, without any sensible error, be d as the same at all points in any such body.

found, by experiment, that the average pressure which the here exerts upon a body, on or near the surface of the s the same as that exerted by a column of water 32 feet in or by a column of mercury 30 inches in height; that is, it with a weight of about 15 lbs. upon every square inch of . That we are not, in ordinary circumstances, cognizant of essure arises from the characteristic property of fluids, in nce with which they press in all directions upon bodies ed in them; so that if the upper surface of a body, subject pressure of the atmosphere, experience a downward pressure lbs. to every square inch, the under surface experiences a onding upward pressure. If, however, we destroy this equi- , by removing or diminishing the pressure of the atmosphere ne side of any body, we are immediately made sensible of ssure which is exerted upon the opposite side. For instance, te of glass be placed upon the top of a cylindrical receiver ed with an air-pump, and the air be exhausted from the r, the plate of glass will be pressed down with a considerable nd if not strong enough will be broken in pieces.

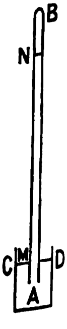
304. *When the atmosphere presses upon any point of a fl*
mass, to determine the amount of transmitted pressure.

If the atmosphere presses upon the highest surface of the flu
mass, then it is evident that the transmitted pressure is equal
the pressure of the atmosphere.

If the atmosphere presses at a point in the fluid mass belo
the highest surface, then a portion of the atmospheric pressu
will be employed in counterbalancing the outward pressure of th
fluid itself, and the remaining portion only of the atmospher
pressure will be transmitted by the fluid. Let h be the distan
below the highest surface of the fluid mass of the point at whic
the atmosphere presses upon it, and let w be the weight of
cubic unit of the fluid; then, as seen in Chapter XVII., the flu
itself will exert at this point an outward pressure on every squa
unit $= wh$. Then if Æ denote the pressure exerted by the atmo
phere upon a square unit, the transmitted pressure will be Æ −
for every unit of surface of the fluid mass.

If the point at which the atmosphere presses upon the flu
mass be so taken that wh is greater than Æ, then it is plain th
equilibrium cannot subsist, but motion will ensue amongst th
particles of the fluid, until the distance becomes such that $wh =$ Æ

305. The Barometer. This instrument is constructed in th following manner:—A glass tube, closed at one end, is filled with mercury, and, a finger being placed over the open end, is inverted in an open vessel of mercury.



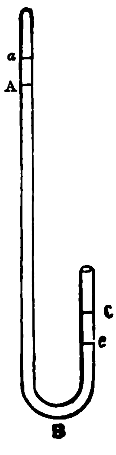
Let AB be such a tube, let CD be the surface of the mercury in the vessel, and let MB be more than 31 inches. Then, since CD is more than 31 inches below B, the highest surface of the fluid mass, the pressure of the fluid upon any square unit in the surface CD will be greater than the atmospheric pressure. The mercury will consequently fall in the tube. Let N be the point at which it rests, then if Æ be the atmospheric pressure, and w the weight of a cubic unit of mercury,

$$Æ = w \, . \, \mathrm{MN}.$$

of mercury in the tube will accordingly vary as the pressure varies; will rise as the atmospheric pressure d fall as it decreases.

ed scale is attached to the tube AB, by means of which in the height of the mercury in the tube can be observed.

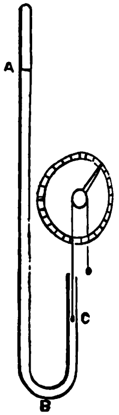
ill be seen, that when the mercury falls in the tube it istern, and that consequently the fall of the mercury as measured by the scale attached to it, will not asure the variation in the barometrical column, but great by the distance through which the mercury has cistern. In like manner, when the mercury rises in variation, as measured by the scale, will be too small ce through which the mercury has *fallen* in the cistern. tern be large in comparison with the bore of the tube, in the height of the mercury in the cistern is so t may be disregarded for ordinary purposes. When racy is desired, the bottom of the cistern is so made e raised or lowered by a screw, and thereby the surface ry in the cistern be adjusted to one uniform height.



SIPHON BAROMETER. In this instrument of any adjustment is avoided by the sim- substituting for the cistern a small tube, bore as that of the larger tube. Let ABC instrument, the surfaces of the mercury nd C. Suppose now the mercury to fall r tube from C to *c*, and in consequence to onger tube from A to *a*. Then, since the tubes is alike, the distance A*a* is equal to tire variation in the barometric column is d therefore equal to 2 A*a*. Hence the measured by the scale, is exactly one-half ariation.

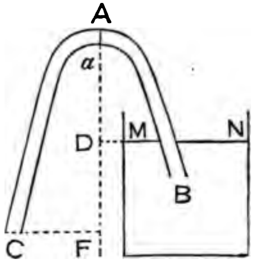
of the instrument possesses some advantages, but is open to the objection that by diminishing the scale it increases the difficulty of observing small variations of the height of the column.

308. THE WHEEL BAROMETER. This form of the barometer has been contrived for the purpose of making small variations of the column more sensible. It consists of a siphon barometer, having a small bar of iron or glass floating upon the open surface of the mercury. A fine thread, attached to the ball, passes round a small wheel carrying an index, which moves over a graduated dial-plate, and the rise or fall of the mercury is measured by the arc of the circle over which the index moves. The variations of the barometric columns range between 28 and 31 inches, or through a distance of three inches. As shown in the last section, the variation at C is one-half of the variation in the barometric column; and, consequently, the ball will rise or fall through a space of 1½ inch. If, then, the circumference of the wheel round which the ring passes be exactly 1½ inch, the index will make a complete revolution when the ball rises or falls through a height of 1½ inch; and if the circumference of the dial-plate be divided into 360 equal parts, the motion of the index over 120 of these parts will correspond to a rise or fall of the ball through half an inch, and therefore to a change of one inch in the barometric column. Consequently, the variation of one 120th part of an inch is shown by the motion of the index over one division of the plate.



309. THE SIPHON. The siphon is a bent tube, open at both ends, and having one leg shorter than the other. If such a tube be filled with any fluid, and placed with the extremity of its shorter leg in a vessel containing the same fluid, the fluid will flow out through the longer leg.

Let ABC be the siphon filled with fluid, and having B, the xtremity of its shorter leg AB, placed elow MN, the surface of the fluid in e vessel. Let A be the highest point the siphon, and through A draw the rtical line AF. Let MN produced eet this vertical line in D, and let CF the horizontal line drawn through C. Let Aa be a thin vertical film of the id passing through A. Since each e of this film is similarly situated with respect to A, the highest nt of the fluid mass, the pressure arising from the weight of the id will be the same on each side. These pressures, therefore, are equilibrium, and do not cause the motion of the fluid.

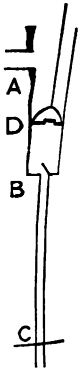


But if Æ be the atmospheric pressure upon a square unit, and w the weight of a cubic unit of the fluid, then, since the atmosphere sses upon the surface MN at a distance AD below the highest nt of the fluid mass, there is transmitted up the leg BA, by t. 304, a pressure upon every square unit equal to $Æ - w . AD$. Similarly, since the atmosphere presses upon the surface of the id mass at C, there is transmitted up CA a pressure upon every are unit equal to $Æ - w . AF$.

Hence, the one side of the film Aa will experience a pressure on every square unit equal to $Æ - w . AD$, and the other side a essure equal to $Æ - w . AF$. But since AD is less than AF, $- w . AD$ is greater than $Æ - w . AF$; the film of fluid will erefore be forced down the leg AC. The same will happen to ery film that may in succession occupy the position Aa; the id consequently will flow out at C, so long as there remains in e vessel any fluid above B.

It has been assumed in the preceding, that the height AD is ch that $w . AD$ is less than Æ. If AD be so great that $w . AD$ greater than Æ, then, as seen in the preceding Articles, the id will sink in the leg AB to some level below the bend, and e siphon consequently will cease to act.

310. THE COMMON PUMP. The common pump consists of cylindrical barrel AB, furnished with a valve at B opening upwards, and with an air-tight piston, capable of being moved up and down by means of a rod.



The piston is also furnished with a valve opening upwards, and to the barrel at B is attached a pipe BC, called the suction pipe.

Let C be the surface of the water to be raised, and suppose the piston to be at B.

Let the piston be raised from B to A; the air in BC will, by virtue of its expansive force, press open the valve at B and fill the barrel AB; and in consequence of occupying a larger space, ABC will exert a less pressure than before upon the surface of the water within the pipe. Hence, the pressure of the air within the pump being less than the atmospheric pressure upon the surface of the water, the water will be forced up the pipe BC until equilibrium be restored.

Let the piston now descend; the valve at B will be closed, and the air in AB will escape through the valve in D. If the piston be again raised, the preceding process will be repeated, and the water will again rise in the pipe BC. In like manner, upon every successive stroke of the piston the water will continue to rise in the pipe, until, if BC be less than 32 feet, it at last reaches B.

Let the water be at B, and let the piston be pressed down to B. Then, since the atmosphere presses upon the surface of the fluid at C, there will be transmitted by the fluid a pressure upon every square unit of the valve at B equal to $Æ - w \,.\, BC$ (Art. 304); and, consequently, when the piston is again raised, the water will force open the valve at B, and will rise in the barrel as far as A, if AC be less than 32 feet, or to some point between B and A, if AC be greater than 32 feet.

Upon the next descent of the piston, the valve in D will be forced open, and the water will flow through it; so that when the piston reaches B the water in the barrel will lie above the piston, d when the piston is raised will be raised along with it, and flow t at the spout A.

It will be seen, that if B be more than 32 feet above C, no ater will rise into the barrel; and in such a case the pump will useless. If B be less, but A more than 32 feet, the water will ly in part fill the barrel, and the discharge of water during the cent of the piston will not be continuous.

311. *In a common pump, whose spout is less than 32 feet from e surface of the water, to determine the resultant pressure upon the ston at any moment during its ascent, after it has commenced disarging the water.*

Let the piston be at D, then, since the water has commenced to discharged, the barrel and pipe are filled with water, and since e piston is rising, the valve at B will be open, and that at D will shut.

The pressure upon the upper side of the piston is the sum of e weight of the fluid mass in AD and the atmospheric pressure on its surface. Therefore, upon every square unit of the upper rface of the piston there is exerted a downward pressure equal to $+ w\,.\,AD$, and upon every square unit of the lower surface of the ston there is, by Art. 304, an upward pressure equal to $\text{Æ} - w\,.\,DC$. t a be the number of square units in the area of the piston,

$\therefore$ the total pressure upon the upper surface $= a\text{Æ} + wa\,.\,AD$,

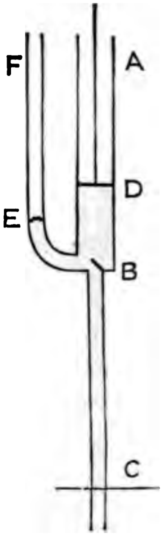
,, ,, lower ,, $= a\text{Æ} - wa\,.\,DC$.

e resultant pressure will be the difference of these pressures, and erefore equals $wa\,.\,AC$.

erefore the resultant pressure is independent of the position of ; in other words, the pressure upon the piston is the same at ery point in its ascent, after commencing to discharge, and is ual to the weight of a column of water, whose base equals the a of the piston, and whose height is the height of the spout of pump above the surface of the water.

312. THE FORCING PUMP. The construction of the forcing pump represented in the accompanying figure.

D is a piston with no valve, working in a barrel AB; BC is a pipe descending to the water; EF a tube leading upwards from B; at B and E are valves opening upwards.



If the piston be raised from B to A, then, as in the common pump, the air in BC will press open the valve at B and fill the barrel, and the water will rise to a certain height in BC. As the piston descends, the air in the barrel will be forced out through the valve at E; after a certain number of ascents the water will, if BC be less than 32 feet, pass into the barrel AB; and, upon its next descent, the piston will force the water in the barrel through the valve E into the pipe EF. As the piston continues to rise, more water will pass into the barrel, and this in its turn will be forced, by the descent of the piston, into the pipe EF.

313. *To determine the resultant pressure upon the piston of forcing pump, at any moment during its ascent or descent, afte it has commenced discharging.*

First. Let the piston be ascending; the valve at B will be open, and that at E will be shut. Then, if Æ be the atmospheric pressure upon a square unit, w the weight of a cubic unit of water, and a the area of the piston, the downward pressure upon the upper surface of the piston will be $a\text{Æ}$, and the upward pressure upon the lower surface of the piston will be $a(\text{Æ} - w\,.\,\mathrm{DC})$, or $a\text{Æ} - wa\mathrm{DC}$. Therefore, the resultant pressure will act downwards, and be equal to $wa\,.\,\mathrm{DC}$.

Hence, since as the piston rises, DC increases, the resultant pressure increases at every moment of the ascent, and at any instant is equal to the weight of a column of water, whose base equals the area of the piston, and whose height is the distance of the piston from the surface of the water C.

Secondly. Let the piston be descending, the valve B will be closed, and that at D will be open. Let F be the highest point in the pipe EF.

The pressure upon the upper surface of the piston will, as before, be a downward force equal to $a\text{Æ}$; and the pressure upon the lower surface of the piston will be an upward force equal to $a\text{Æ} + wa \,.\, \text{DF}$. The resultant pressure will therefore be an upward force equal to $wa \,.\, \text{DF}$.

Hence, the pressure upon the piston increases at every moment of the descent, and at any instant equals the weight of a column of water, whose base equals the area of the piston, and whose height is the distance of the piston below the point at which the pipe discharges.

EXAMPLES.

1. The average height of the barometer at Paris, as obtained from a large number of observations, is 29·77 inches; determine the average pressure of the atmosphere. Ans. 14 lbs. 10·3 oz.

2. A siphon is used for decanting a liquid whose specific gravity is 1·7; determine the limit to the height of the highest point of the siphon above the surface of the fluid, when the barometer stands at 29 inches. Ans. 19 ft. 4 in.

3. The area of the piston of a common pump is 20 inches, and the height of the spout above the surface of the water is 24 feet; determine the pressure upon the piston. Ans. 208 lbs. $5\frac{1}{3}$ oz.

4. If a pump be used to raise a fluid whose specific gravity is 1·5; find the limiting height of the bottom of the barrel, when the barometer stands at 29·5 inches. Ans. 22 ft. $3\frac{1}{2}$ in. nearly.

5. If two vessels, containing fluids of different specific gravities which do not mix, be placed with their surfaces in the same horizontal plane; and if a siphon, having its legs filled respectively with the two fluids, be placed in the vessels, so that each leg is in the same fluid as that with which it is filled; show that the lighter

fluid will force the heavier one entirely down the leg of the siphon, supposing that the level in either vessel is not perceptibly altered by the addition or removal of as much fluid as is contained in one leg of the siphon.

6. Two vessels, containing fluids of different specific gravities which do not mix, are placed so that the surfaces of the fluids are at different levels, and a siphon, having its legs filled respectively with the two fluids, is placed in the vessels with each leg in the same fluid as that with which it is filled; determine the position of the common surface of the two fluids, supposing, as before, that the level in either vessel is not appreciably altered by the addition or removal of as much fluid as is contained in one leg of the siphon.

Let s_1 and s_2 be the specific gravities of the fluids, and h_1 and h_2 the distances of the highest point of the siphon from the surfaces of the fluids; then the distance of the common surface below the highest point of the siphon is equal to

$$\frac{s_1h_1 - s_2h_2}{s_1 - s_2}.$$

7. Let the two vessels in example 5 be cylindrical and with equal bases, determine the position of the common surface of the two fluids when the alteration in the levels is not disregarded; also determine the amount of alteration in the levels, the siphon being of uniform bore.

Let s_1 and s_2 be the specific gravities of the two fluids, s_1 being that of the heavier. Let A be the area of the base of the vessels, and a that of a section of the siphon. Let h be the distance of the highest point of the siphon above the original level. Then the distance of the common surface below the highest point of the siphon is

$$\frac{(s_1 - s_2)\,\mathrm{A}h}{\mathrm{A}\,(s_1 - s_2) + a\,(s_1 + s_2)},$$

and the alteration of level is equal to

$$\frac{(s_1 - s_2)\,ah}{\mathrm{A}\,(s_1 - s_2) + a\,(s_1 + s_2)},$$

the lighter fluid being depressed, and the heavier fluid being raised by this amount.

CHAPTER XX.

ON THE LAWS OF ELASTIC FLUIDS.

ł. A gas or aëriform fluid is distinguished from other fluids e tendency of its particles to recede from each other; that is, ı excess of the repulsive forces over the attractive forces ; upon its particles. If such a fluid be enclosed in any , it will, like any other fluid, exert a pressure upon the sides e vessel in consequence of the action of gravity upon its les; but, irrespectively of this, it will exert another, and only far greater pressure, in consequence of the mutual re- n of its particles. It is this latter pressure which is termed *astic force* of the gas. This is found to vary with the space ı which the gas is enclosed, and also with the temperature. ıw of this variation, as approximately determined by experi- is exhibited in the following articles.

i. LAW I. *The temperature remaining the same, the elastic of any gas varies inversely as the space it occupies.*

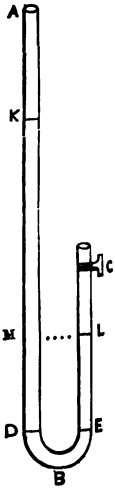
is law is often quoted as "Boyle and Mariotte's law," or y as Boyle's law.

e experiments by which this law is established are of the ing kind.

glass tube of uniform bore, open at both ends, is bent into the ABC; the shorter branch BC is furnished with a stop-cock C, n accurately divided scale is attached to the longer branch.

; the stop-cock be opened, and a small quantity of mercury ured into the tube. Let the surfaces of the mercury be at 1 E; these will be in the same level, since the same pressure, he atmospheric pressure, acts upon both. Let the stop-cock now be closed, the surfaces D and E will remain at the same level, and the pressure of the air in CE upon the surface at E is equal to the pressure of the atmosphere upon D; or the elastic force of the air in CE is equal to the atmospheric pressure. Let the barometer stand at h inches, and let w be the weight of a cubic inch of the mercury used in the barometer, then the atmospheric pressure per inch, and consequently the elastic force of the air in CE, is equal to

$$wh.$$



Now let mercury of the same temperature and density as that in the barometer be poured into the larger branch, and let the mercury rise to L in the shorter branch, and stand at K in the longer branch; then the pressure at L must be equal to the pressure at M. The pressure at M per inch is the atmospheric pressure increased by the weight of KM inches of mercury; hence, the elastic force of the air in CL is equal to,

$$wh + w\,.\,\text{KM}.$$

Hence we obtain this result,

$$\frac{\text{elastic force of the air in CL}}{\text{elastic force of the air in CE}} = \frac{wh + w\text{KM}}{wh},$$

$$= \frac{h + \text{KM}}{h}.$$

It is found by trial that

$$\frac{\text{CE}}{\text{CL}} = \frac{h + \text{KM}}{h};$$

and therefore it follows that

$$\frac{\text{elastic force of air in CL}}{\text{elastic force of air in CE}} = \frac{\text{CE}}{\text{CL}},$$

or the elastic force of the air varies inversely as the space it occupies.

By similar experiments the law is established for other gases.

Hence, if p_1 be the elastic force, and V_1 the volume of a given ass of gas at a given temperature, and if p_2 be the elastic force ien the same mass has become compressed or expanded into the lume V_2, the temperature remaining the same; then

$$p_1 : p_2 : : V_2 : V_1,$$

or,

$$V_1 p_1 = V_2 p_2.$$

In performing the experiments described above, care must be ken to guard against any increase of temperature by the sudden mpression of the gas. The mercury should be poured in very owly, and some time allowed before observing the levels, that ıy heat which may have been generated may pass away by diation.

316. Since the density of any body varies inversely as the space occupies, it follows that Boyle and Mariotte's law may be thus unciated. *The temperature remaining the same, the elastic force any gas varies directly as the density.*

317. The following are examples in illustration of Boyle's law:—

Ex. 1. Determine the elastic force of a mass of air, whose ume is 100 cubic inches, when compressed into 48 cubic inches; barometer standing at 30 inches.

The elastic force of the air before compression is equal to the iospheric pressure, that is, to the weight of 30 inches of mercury, 14·757 lbs. Hence, if p be the elastic force required,

$$p = \frac{100 \times 14{\cdot}757}{48},$$

$$= 30{\cdot}744 \text{ lbs.}$$

Ex. 2. An air-tight piston is fitted into a cylindrical vessel of form bore, closed at one end; the vessel is filled with air, and piston forced in with a pressure of 50 lbs. The length of the inder is 20 inches, the area of the piston 5 square inches, and barometer stands at 29 inches; how far will the piston be ed in?

The elastic force of the air before compression is equal to the atmospheric pressure, that is, to the weight of 29 inches of mercury, or 14·265 lbs.

The elastic force of air after compression is equal to the atmospheric pressure increased by the applied pressure; the latter is $\frac{50}{5}$, or 10 lbs., hence the elastic force is 24·265.

Then if x be the distance through which the piston is forced, the reduced volume is to the original volume as $20-x : 20$; and therefore

$$\frac{20-x}{20}=\frac{14{\cdot}265}{24{\cdot}265};$$

whence,

$$x=\frac{200}{24{\cdot}265}$$
$$=8{\cdot}243 \text{ inches.}$$

318. Law II. *The pressure remaining the same, equal increments of temperature will cause in any gas equal increments of volume.*

This is called Dalton and Gay Lussac's law.

If V_0 be the volume of any gas at the freezing point, and aV_0 be the increment caused by an increase of one degree in the temperature, then $a\theta V_0$ is the increment caused by θ degrees; and hence, if V be the volume of the gas at the temperature of θ degrees above the freezing point.

$$V=V_0+a\theta V_0$$
$$=V_0(1+a\theta).$$

The quantity a is called the *co-efficient of expansion*, and for all gases its value is $\frac{1}{273}$ for the Centigrade scale, or $\frac{1}{491}$ for the Fahrenheit scale. Hence, if τ be the temperature on the Centigrade scale

$$V=V_0 . \frac{273+\tau}{273}.$$

If τ be the temperature on the Fahrenheit scale

$$V=V_0 . \frac{491+(\tau-32)}{491}=V_0 . \frac{459+\tau}{491}.$$

319. *To compare the volumes of a given quantity of gas at different temperatures, but under the same pressure.*

τ_2 be the temperatures on the Centigrade scale, and let V_1, ie corresponding volumes of the gas. Then, by the pre- f V_0 be the volume at the freezing point,

$$V_1 = V_0 \cdot \frac{273+\tau_1}{273},$$

$$V_2 = V_0 \cdot \frac{273+\tau_2}{273};$$

, $$\frac{V_1}{V_2} = \frac{273+\tau_1}{273+\tau_2}.$$

mperatures be given on the Fahrenheit scale,

$$\frac{V_1}{V_2} = \frac{459+\tau_1}{459+\tau_2}.$$

To compare the volumes of a given quantity of gas at temperatures and under different pressures.

$_1$ be the volume of the gas, at a temperature θ_1 under a p_1, and let V_2 be the volume of the same quantity of gas perature θ_2, under a pressure p_2. Let the temperatures in s be reduced to the freezing point; then, by Art. 318, the are respectively

$$\frac{V_1}{1+a\theta_1}, \text{ and } \frac{V_2}{1+a\theta_2}.$$

e the temperatures are the same, the pressures are inversely lumes (Law i.), consequently

$$p_1 : p_2 :: \frac{V_2}{1+a\theta_2} : \frac{V_1}{1+a\theta_1},$$

$$\frac{V_1 p_1}{V_2 p_2} = \frac{1+a\theta_1}{1+a\theta_2}.$$

when the temperature is given by the Fahrenheit ther-

, $$\frac{V_1 p_1}{V_2 p_2} = \frac{459+\tau_1}{459+\tau_2};$$

e Centigrade thermometer,

$$\frac{V_1 p_1}{V_2 p_2} = \frac{273+\tau_1}{273+\tau_2}.$$

If the zero point of the thermometer be taken at $-273°$
$-459°$ F.), then the temperatures as given by the therm
will represent the values, $273+\tau_1$, $273+\tau_2$ (or $459+\tau_1$, 45
used in the preceding expressions. This point is called the a
zero, and temperatures reckoned from this zero are called th
lute temperatures. Hence the result of the preceding investi
may be thus enunciated: *For any gas, the product of the*
and pressure varies directly as the absolute temperature.

321. If ρ_1, ρ_2 denote the densities of the gas in the two
then, since the densities are inversely as the volumes, the re
the preceding section becomes

$$\frac{\rho_2}{\rho_1}=\frac{p_2(1+a\theta_1)}{p_1(1+a\theta_2)},$$

or,

$$\frac{p_1}{p_2}=\frac{\rho_1(1+a\theta_1)}{\rho_2(1+a\theta_2)},$$

which gives us the relation between the pressure, temperatu
density of any given quantity of gas.

The form of this result may be conveniently modified as f
Let ρ be the density at the freezing point under a standar
sure p. Then

$$\frac{p_1}{p}=\frac{\rho_1(1+a\theta_1)}{\rho},$$

or,

$$p_1=\frac{p}{\rho}.\rho_1(1+a\theta_1).$$

But $\frac{p}{\rho}$ is a constant whose value may be determined by
ment for each particular gas. Let k denote the constan

$$p_1=k\rho_1(1+a\theta_1);$$

or, as it may be written,

$$p=k\rho(1+a\theta).$$

322. The following are examples in illustration of Dalt
Gay Lussac's law.

Ex. 1. Required the increase in the volume of a cubic
air, when the temperature is raised from 51° F. to 71°
pressure remaining unchanged.

Let V denote the increased volume; then, by Art. 319,

$$V = \frac{459 + 71}{459 + 51}$$
$$= \frac{530}{510}$$
$$= 1\tfrac{2}{51} \text{ cubic feet.}$$

Therefore the required increase is $\frac{2}{51}$ cubic feet, or 67·76 cubic inches.

Ex. 2. If a certain quantity of air have a volume of 100 cubic nches when the barometer stands at 30 inches, and the temperaure is 51° F.; determine its volume when the barometer is at 9 inches, and the temperature 71° F.

Let V be the volume required, then, by Art. 320,

$$\frac{V}{100} = \frac{30}{29} \cdot \frac{459 + 71}{459 + 51} = \frac{30}{29} \cdot \frac{530}{510};$$

$$\therefore \qquad V = 107·5 \text{ cubic inches.}$$

Ex. 3. If 100 cubic inches of air weigh 31 grains when the rometer is at 30 inches and the temperature 60° F.; determine e volume of 2 oz. of air when the barometer is at 29 inches and e temperature is at 122° F.

Since 100 cubic inches weigh 31 grains, $\frac{100 \times 875}{31}$ inches will igh 875 grains, or 2 oz. Then, if V be the required volume, by rt. 320,

$$\frac{31V}{87500} = \frac{30}{29} \cdot \frac{459 + 122}{459 + 60};$$

$$\therefore \qquad V = 3270·9 \text{ cubic inches.}$$

323. Law iii. *When two gases, which do not chemically unite, e mixed together, each gas acts as a vacuum with respect to the her.*

If two vessels, containing two different gases which do not emically unite, be brought together, and a communication be ened between them, each gas is found, after a short interval, to equally diffused through both vessels. This result is independent of the specific gravities of the two gases, so that if the heavier gas be in the lower vessel, and the lighter one in the higher, the heavy gas will rise into the higher vessel, and the lighter gas will descend into the lower vessel.

In stating that each gas acts as a vacuum with regard to the other, Dr. Dalton, who first enunciated this law, did not mean to imply that the diffusion of one gas through the space occupied by another takes place with the same rapidity as if there had been an actual vacuum, but that the final result is the same.

324. *Given the volumes and elasticities of two gases to find the elastic force of the mixture.*

Let V_1 be the volume, and p_1 the elastic force of the gas A; and V_2 the volume, and p_2 the elastic force of the gas B.

When intermixture takes place, the gas A acquires the volume $V_1 + V_2$; and hence, by Law i.,

$$\text{the elastic force of A} = \frac{V_1 p_1}{V_1 + V_2}.$$

In like manner,

$$\text{the elastic force of B} = \frac{V_2 p_2}{V_1 + V_2};$$

but the total pressure on any unit of the containing surface is the sum of the pressures exerted severally by the two gases. Hence,

$$\text{elastic force of the mixture} = \frac{V_1 p_1 + V_2 p_2}{V_1 + V_2}.$$

Cor. i. If the two gases before mixture have the same elastic force, that is, if $p_2 = p_1$, then

$$\text{elastic force of the mixture} = \frac{V_1 p_1 + V_2 p_2}{V_1 + V_2} = p_1,$$

or the elastic force is unaffected by the mixture.

Cor. ii. If the volumes of the two gases be equal, or $V_2 = V_1$, then

$$\text{elastic force of the mixture} = \tfrac{1}{2}(p_1 + p_2).$$

EXAMPLES.

1. An air-tight piston is fitted into a cylindrical vessel filled th air; the length of the cylinder is 30 inches, and the area of e piston is 10 inches; required the force necessary to push the ton in through 15 inches, the barometer standing at 30 inches.

Ans. 147·57 lbs.

2. In the preceding, determine the force required to draw the ton out through 15 inches. Ans. 49·19 lbs.

3. If the pressure of P lbs. will force a piston, whose area is a hes, into a cylindrical vessel containing air through a distance c inches when the barometer stands at h_1 inches, how far will same pressure force in the piston when the barometer stands h_2 inches?

Let $\frac{P}{a}$ be equal to the weight of h inches of mercury, then the uired distance is equal to $\frac{c(h+h_1)}{h+h_2}$.

4. If 100 cubic inches of gas at freezing point be raised to 53° what will be the volume? Ans. 119·5 cubic inches.

5. If 100 cubic inches of gas at 27° C. be lowered in temperature the freezing point, what will be the volume?

Ans. 91 cubic inches.

6. If 20 cubic inches of gas at 57° F. be heated to 109° F., ermine the increase of volume. Ans. 2·015 inches.

7. If 142 cubic inches of gas at 109° F., cool down to 53°, ermine the loss of volume. Ans. 14 inches.

8. A strong vessel is filled with air at a temperature of 41° F., heated to 212° F.; what is the elastic force of the enclosed air, barometer standing at 30°? Ans. 19·804 lbs.

9. Assuming that 100 cubic inches of air weigh 31 grains the barometer stands at 30 inches and the thermometer at determine the weight of a cubic foot of air when the barom at 30·5 inches, and the thermometer at the freezing point.

Ans. 575·66 gra

10. A cylindrical vessel, closed at the top, is sunk in wate given depth, determine the height to which the water rises i vessel.

Let a be the height of the vessel, and h_1 the depth of the of the vessel below the surface of the water. Let the atmos pressure be equal to the weight of a column of water whose is h. Then, if w be the weight of a cubic unit of wate atmospheric pressure is equal to

$$wh.$$

Let x be the height to which the water rises in the vessel, the pressure at the surface of the water in the vessel, that is, depth of $h_1 - x$, is equal to

$$wh + w(h_1 - x),$$

which gives us the elastic force of the compressed air. Bu Law i., the elastic force is inversely as the space; therefore

$$\frac{wh + w(h_1 - x)}{wh} = \frac{a}{a - x},$$

or,

$$\frac{h + h_1 - x}{h} = \frac{a}{a - x}.$$

Whence, by solving this quadratic, we obtain

$$x = \tfrac{1}{2}(a + h + h_1) - \tfrac{1}{2}\sqrt{\{(a + h + h_1)^2 - 4ah_1\}}.$$

11. A diving bell, of cylindrical form, is sunk in water to a depth, how much must the temperature of the enclosed ai raised that the water may not rise in the diving bell?

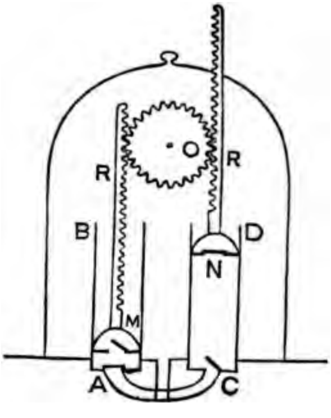
Let the temperature of the air when first enclosed be Let h_1 be the depth of the bottom of the diving bell, and height of a column of water equal to the atmospheric press then the required increase of temperature will be equal to

$$\frac{h_1(459 + \tau)}{h}.$$

CHAPTER XXI.

ON THE AIR-PUMP AND STEAM-ENGINE.

[HE DOUBLE-BARREL AIR-PUMP. AB and CD are two al barrels connected together by the pipe AC, and com- ng by means of another pipe with an air-tight receiver R. [are pistons furnished with valves opening upwards, and ed by the toothed wheel O, so that when M descends N ıd, and *vice versa.* At A and C are valves opening upwards. be at A, and N at D, and the wheel be turned so that N end and M ascend. As N descends, the valve at C will d the air in the barrel pass out through the N. As M ascends, the of the external air will valve in M; and the being removed from the le of the valve A, the ıe receiver will, in con- of its expansive force, n the valve and fill the B. The air originally ceiver, now occupying a ıce, viz. the receiver and ether, will be of dimin- ısity.



wheel be again turned, a similar process will take place, ıir in the receiver will again expand, so as to fill both ınd barrel, and a second diminution of density will conse- ake place.

The process may be continued so long as the air in the rece can by its expansive force press open the valves at A or C; since the expansive force of the air is diminished with the dim tion of its density, the action of the pump must, after a cei number of strokes, entirely cease.

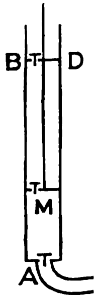
326. The Single-barrel Air-Pump. This air-pump consis a single barrel AB, communicating with an air-tight receive the pipe C.

The piston passes through an air-tight collar in the plate At B is a valve opening upwards.

The process of exhaustion is precisely similar to that descr in Art. 325.

The effect of the valve at B is to relieve the piston from pressure of the atmosphere during a part of its ascent. For if the air in BM be of less density than atmospheric air, the pressure of the atmosphere will press down the valve at B, and will keep it closed until the piston by its ascent has so compressed the air in BM that its density has become greater than that of atmospheric air.

The plate BD, with its valve opening upwards, is not peculiar to the single-barrel air-pump, but may, if required, be adopted in a double-barrel air-pump. In fact, it is used in the more carefully-constructed double-b pumps.



327. *To find the density of the air in the receiver of an pump after any number of strokes of the piston.*

If a quantity of air expand so as to occupy a space twi great as before, its density will evidently be one-half its ori density; if the space be three times as great, the density wi one-third the original density; and so, generally, the density expansion will be to the density before expansion as the occupied before expansion is to the space occupied after expan

Hence, if A be the content of the receiver, and B of each o rels, since at every stroke the air in the receiver expands, and s both receiver and barrel,

$$\frac{\text{lensity of the air at the close of any stroke}}{\text{lensity of the air at the commencement of the stroke}} = \frac{A}{A+B}.$$

nsequently,

$$\frac{\text{density of the air after 1 stroke}}{\text{density of atmospheric air}} = \frac{A}{A+B}.$$

0,

$$\frac{\text{density of the air after 2 strokes}}{\text{density of the air after 1 stroke}} = \frac{A}{A+B},$$

l $\therefore$

$$\frac{\text{density of the air after 2 strokes}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^2.$$

like manner,

$$\frac{\text{density of the air after 3 strokes}}{\text{density of the air after 2 strokes}} = \frac{A}{A+B},$$

l $\therefore$

$$\frac{\text{density of the air after 3 strokes}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^3.$$

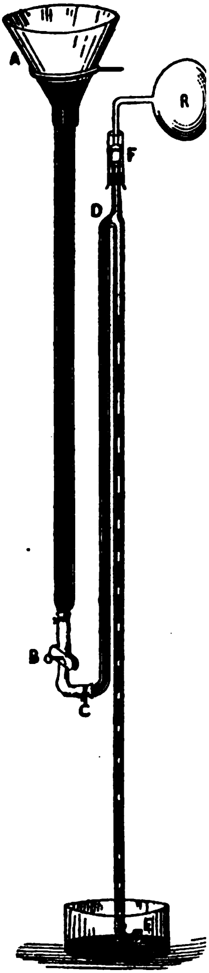
Hence, generally,

$$\frac{\text{density of the air after } n \text{ strokes}}{\text{density of atmospheric air}} = \left(\frac{A}{A+B}\right)^n.$$

This result shows that, even if there were no practical limits to working of an air-pump, the air could never be entirely exsted; for the density of the air in the receiver always has some ignable ratio to the density of atmospheric air.

28. THE SPRENGEL AIR-PUMP. By means of this instrument uch nearer approach to a perfect vacuum can be attained than ossible with the common air-pump. It consists of a long glass e, BCDE, bent as in the figure, and having the limb CD longer n the barometric column. The bore of the longer arm DE and the upper part of CD is very fine, about one-tenth of an inch diameter. At B the tube is connected by means of a piece of vas-covered india-rubber tubing with the pipe of the funnel A. nection between the funnel and the tube CD can be closed by ns of a clamp at B. At D is a short branch tube connected the india-rubber tubing F with the receiver R.

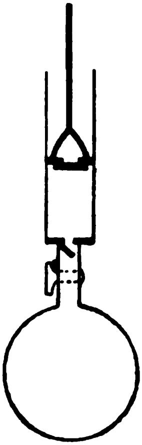
When the instrument is in action the funnel A is fille with mercury, and the clamp at B being loosened, the mercury rises in CD, and falls down DE in successive drops. As the mercury passes the opening of the branch tube at D it carries with it a portion of the air from the receiver, and DE becomes filled with a number of columns of mercury separated by small columns of air. As DE is longer than the barometric column for mercury, the mercury, and with it the enclosed air, will run out at E into a vessel placed there to receive it.



As the exhaustion proceeds the number of columns of air become fewer and fewer, until at length the lower part of the tube DE shows a continuous column of mercury nearly equal in height to the barometric column. When this is so the exhaustion is complete, no more globules of air being enclosed within the mercury. The time required for reaching this point varies of course with the size of the receiver. If the receiver be about a pint in volume from 20 to 30 minutes is required. The instrument itself gives notice to the operator of the approaching completion of the operation by the sharp metallic noise which result when a liquid falls in a vacuum.

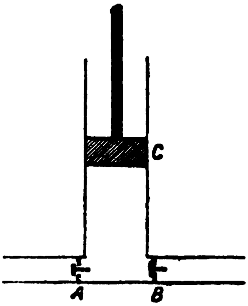
By means of the Sprengel pump the density of the air in the receiver can be reduced to less than one millionth part of that of the atmosphere.

329. **The Condensing Air-Pump.** This differs from the single-arrel air-pump only in having the two valves opening towards the ceiver instead of away from it. As the piston escends the valve in the piston becomes closed, nd the valve at the bottom of the barrel opens, nd the air in the barrel is forced into the receiver. Vhen the piston is raised, the elastic force of the r in the receiver closes the valve in the barrel, d so prevents the egress of the contained air. he pressure of the atmosphere opens the valve in e piston, and the barrel is thus a second time filled th air.



It will be seen that by each descent of the ton a quantity of air is forced into the receiver ual, in its original volume, to the content of the rel.

By a simple arrangement, the same barrel may be de available as both an exhausting and a condensing syringe. e piston C is solid. Two tubes A and B furnished with ves lead from the bottom of the rel, the valves both opening in the ne direction (as represented in the ure both open to the left). If the e A be connected with the receiver syringe will act as a condensing inge, the air entering at B during *upward* stroke of the piston. If the e B be connected with the receiver, syringe will act as an exhausting inge, the air being expelled through during the *downward* stroke of the piston.

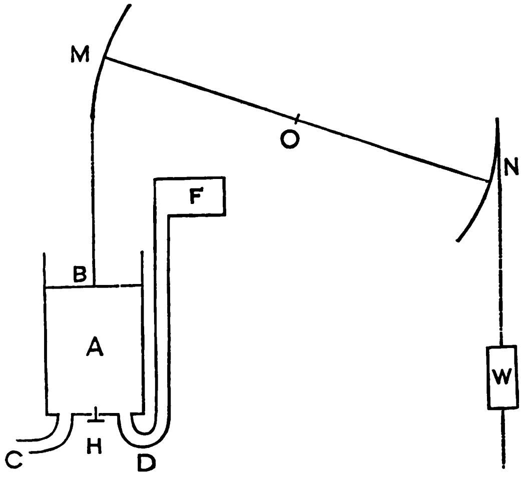


330. **The Atmospheric Steam-Engine.** B is a solid piston rking in the hollow cylinder A, and connected with one ex- mity of the beam MN by a chain attached to the piston-rod.

C is a pipe leading from the boiler, and D a pipe leading from a reservoir of cold water F; C and D are both furnished with stop-cocks. H is a valve opening downwards.

The beam MN moves about its centre O, and to the extremity N is attached a weight or counterpoise W, equal to half the atmospheric pressure upon the surface of the piston B.

Suppose the piston B to be at the bottom of the cylinder, and let the pressure exerted by the steam in the boiler be equal to that exerted by the atmosphere. If the cock in C be now opened, the steam will enter the cylinder, and will press upon the under surface of the piston with a force equal to the atmospheric pressure upon the upper surface. The weight W will consequently descend, and by its descent will raise the piston B to the top of the cylinder.



Let the cock in C be now closed, and that in D be opened; a jet of cold water will issue into the cylinder, and condensing the steam in A will leave a vacuum below B. Then, since the pressure of the atmosphere upon B is double the weight W, B will be pressed down with a force equal to W.

When B arrives at the bottom of the cylinder, the cock in D is closed, and that in C is opened, and the piston ascends as before.

The water in the cylinder escapes by the valve H., which is forced open by the pressure of the steam when first admitted, and is conveyed by a pipe into a cistern communicating with the boiler.

331. THE SINGLE-ACTING STEAM-ENGINE. This engine differs from that described in the preceding article in employing, instead of atmospheric pressure, the pressure of steam to impel the piston downwards. It is thus in the strictest sense a *steam*-engine.

The top of the cylinder is closed, with the exception of a passage for the piston-rod. This passage is furnished with a stuffing-ox filled with tow saturated with grease, so that while the piston-od can move freely up and down, the steam is prevented from scaping. The steam is admitted into the cylinder above the iston, at the moment that a vacuum has been produced below ie piston by condensation; the pressure of the steam then forces ie piston downwards. When the piston has arrived at the lowest oint, the steam which fills the cylinder above the piston is allowed circulate below the piston (a communication between the upper id lower parts of the cylinder being opened for this purpose). he piston is then pressed equally by the steam upon its upper id lower surfaces, and will consequently be raised by the weight tached to the beam.

In the single-acting steam-engine the pressure of the steam— in the atmospheric engine the pressure of the atmosphere— ts only during the descent of the piston; and in both engines ie force necessary to raise the counterpoise was so much subacted from the available power of the engine. These objections ere obviated by the invention of the double-acting steam-engine, which the upward as well as the downward stroke was caused y the pressure of the steam.

332. THE DOUBLE-ACTING STEAM-ENGINE. AB is a hollow ylinder closed at both ends. M is a solid piston. The piston-d passes through a stuffing-box at B, and is connected with one

part of the cylinder and the atmosphere. The steam in the upper part of the cylinder will escape into the atmosphere, and the pressure upon the lower surface of the piston being greater than that upon its upper surface, the piston will ascend.

Non-condensing engines are popularly, but improperly, called *high-pressure steam-engines,* and, in contradistinction, the condensing engines are called *low-pressure steam-engines.*

MISCELLANEOUS EXAMPLES.

1. The arms of a lever of the first kind, of uniform thickness and density, are a and b, the weight of a unit of the bar is m, what weight must be attached to the extremity of the shorter arm that it may balance the longer arm?

The weight required is equal to $\frac{m}{2} \cdot \frac{a^2 - b^2}{b}$.

2. If the inclination of a plane be 30°, show that the time of descent is equal to that of a body falling freely through twice the length of the plane.

3. If a solid body, whose specific gravity is s, be placed in a vessel containing two fluids that do not mix, whose specific gravities are respectively s_1 and s_2, s_1 being greater and s_2 being less than s, show that the part immersed in the heavier liquid is to the whole solid as $s - s_2 : s_1 - s_2$.

4. If a triangle, whose height is h, be divided by a line drawn parallel to its base, at a distance from the vertex of one nth of the total height, what is the distance of the centre of gravity of the lower portion from the base of the triangle?

The distance required is equal to $\frac{h}{3}\left(1 - \frac{2}{n(n+1)}\right)$.

5. If a pyramid, whose height is h, be divided by a plane drawn parallel to its base at a distance from the vertex of one nth of

the total height, what is the distance of the centre of gravity of the frustum from the base of the pyramid?

The distance required is equal to $\frac{h}{4}\left(1 - \frac{3}{n(n^2+n+1)}\right)$.

6. If a cone be immersed in any fluid so as to be just covered, show that the total pressure upon the conical surface, when the vertex is upwards, is twice as great as when the vertex is downwards.

7. Two cords, bearing equal weights P, P, and passing over two pulleys A, B, which lie in the same horizontal line, at a distance from each other equal to $2a$, are attached at a point C to a weight W, what in the position of equilibrium is the distance of C below the horizonal line through A and B?

The distance required is equal to $\frac{Wa}{\sqrt{(4P^2 - W^2)}}$.

8. If the side of a vessel filled with fluid be a rectangle, and it be divided by a horizontal line drawn at one nth of the depth; show that the pressure on the lower portion is to that on the upper as $n^2 - 1 : 1$.

9. Two fluids that do not mix, and whose specific gravities are s_1, s_2, are poured into a small bent tube of uniform bore, in such quantities as would fill respectively a and b inches of the tube; what is the distance of the common surface of the fluids from the bottom of the tube?

The distance required is $\frac{as_1 - bs_2}{2s_1}$ if as_1 be greater than bs_2, but $\frac{bs_2 - as_1}{2s_2}$ if bs_2 be greater than as_1.

10. If a, b, c be the sides of a triangle, and k, l, m the lines rawn from the centre of gravity to the vertices; show that $^2 + b^2 + c^2 = 3(k^2 + l^2 + m^2)$.*

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* Before attempting this or the next example, prove the following theorem y aid of the Second Book of Euclid. If in any triangle ABC a point D be ken in the base AB, such that DB is one nth part of AB, then

$$CD^2 = \frac{n \cdot AC^2 + n(n-1) \cdot BC^2 - (n-1)AB^2}{n^2}.$$

11. If a, b, c, d, e, f be the edges of a triangular pyramid, and k, l, m, n the distances of the centre of gravity from each vertex; show that

$$a^2+b^2+c^2+d^2+e^2+f^2=4\,(k^2+l^2+m^2+n^2).$$

12. If a triangular plate float vertically in any fluid, show that, in the position of equilibrium, the lines joining the bisection of that side which is either wholly within or wholly without the fluid, with the points at which the other sides meet the surface of the fluid, will be equal to one another.

13. The side of a vessel filled with any fluid is a triangle, having its vertex at the bottom of the vessel, and is divided by a horizontal line drawn at one nth of the depth; show that the pressure on the whole side is to that on the lower portion as

$$n^3 : (n-1)^2\,(n+2).$$

14. A common pump is worked by a lever, whose arms are p and q, the spout is h inches above the surface of the water; what must be the area of the piston that the pump may be worked by a power P?

The area of the piston is $\dfrac{\mathrm{P}p}{qh}\cdot\dfrac{1728}{1000}$ inches.

15. A body is projected with a velocity V up a smooth inclined plane, rising h in l; determine the length of the ascent.

The required distance is equal to $\dfrac{\mathrm{V}^2 l}{2gh}$.

16. An inelastic body falling on to an inclined plane from any point in the perpendicular to the plane drawn through its highest point, will reach the bottom of the plane in the same time, and with the same velocity, as a body falling down the whole length of the plane.

17. A beam of uniform thickness, whose length is l, and weight W, is placed with one end upon a horizontal plane, and the other

on the summit of an inclined plane, whose height is h and base ; show that the horizontal force to be applied at the foot of the am to keep it at rest is

$$\frac{W}{2} \cdot \frac{h\sqrt{(l^2-h^2)}}{h^2+b\sqrt{(l^2-h^2)}}.$$

18. A triangular plate ABC, having the angle ACB a right gle, is fastened by a pivot at C, and has weights P and Q sus-nded from A and B; show that, in the position of equilibrium, AC $= a$, and BC $= b$, and if x and y be the distances of A and B verally below the horizontal line drawn through C,

$$x = \frac{a^2 P}{\sqrt{(a^2 P^2 + b^2 Q^2)}} \text{ and } y = \frac{b^2 Q}{\sqrt{(a^2 P^2 + b^2 Q^2)}}.$$

19. If, in the preceding, ABC be an equilateral triangle, whose de is a, prove that in the position of equilibrium,

$$x = \frac{a}{2} \cdot \frac{2P+Q}{\sqrt{(P^2+Q^2+PQ)}} \text{ and } y = \frac{a}{2} \cdot \frac{2Q+P}{\sqrt{(P^2+Q^2+PQ)}}.$$

20. If a straight line be drawn from the highest point of a given rcle to a given point within it, and be produced to meet the rcle, the produced part is the line of quickest descent from the iven point to the circle.

21. If h_1, h_2, h_3 be the distances of the vertices of a triangle om any plane, show that the distance of the centre of gravity, om the same plane, is equal to

$$\tfrac{1}{3}(h_1 + h_2 + h_3).$$

22. The two parallel sides of a trapezium are at right angles ith the base; required the distance of the centre of gravity from e base.

Let h_1 and h_2 be the lengths of the two parallel sides, then the stance required will be equal to

$$\frac{1}{3} \cdot \frac{h_1^2 + h_1 h_2 + h_2^2}{h_1 + h_2}.$$

23. If equal weights of three substances, whose specific gravities are s_1, s_2, and s_3 be compounded, without any change of volume, and if s be the specific gravity of the compound, show that

$$\frac{1}{s} = \frac{1}{3}\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3}\right).$$

24. A body composed of n different substances, whose volumes are V_1, V_2, and specific gravities s_1, s_2, &c., just floats in water; show that

$$V_1(1 - s_1) + V_2(1 - s_2) + \&c. \ldots\ldots + V_n(1 - s_n) = 0.$$

25. A body whose weight is W, composed of two substances, A and B, displaces, when entirely immersed, V inches of water; corresponding weights of A and B displace V_1 and V_2 inches: determine the weight of A contained in the compound.

The weight required is equal to

$$W \cdot \frac{V - V_2}{V_1 - V_2}.$$

26. If 100 cubic inches of air weigh 31 grains, show that the weight of air contained in a globe one foot in diameter is very accurately one 25th part of a pound avoirdupois.

27. If the volume of A is m times the volume of B, and the specific gravity of A is n times the specific gravity of B, show that the weight of A is mn times the weight of B.

28. The content of the receiver of an air-pump is 5 times that of the barrel; determine the elastic force of the air after 10 strokes of the piston, the atmospheric pressure being estimated at 15 lbs. to the square inch.

The elastic force is 2·4226 lbs.

29. A barometer standing at 30 inches is placed under the receiver of an air-pump; at what height will the mercury stand after 12 strokes of the piston, the content of the receiver being 9 times that of the barrel?

The required height is 8·4728 inches.

30. What volume of water must be added to a pint of fluid hose specific gravity is ·52 that the specific gravity of the mixture ay be ·68?

The required quantity of water is half-a-pint.

31. A body falls from the highest to the lowest point of a circle hose plane is vertical along two of the sides of the inscribed quare; supposing that there were no loss of velocity in passing rom the first side to the second, show that the time of descent ould be twice that of falling freely down the radius of the circle.

32. A smooth hollow tube ABC, having the parts AB and BC qual, and bent so that the angle ABC is equal to 120°, is placed ith its lower leg BC vertical; determine the time in which a ball ust fitting the tube will fall from A to C.

Let a be the length of each of the legs of the tube, then the time equired is equal to

$$3\sqrt{\left(\frac{a}{g}\right)}.$$

33. In the preceding, if the upper leg be vertical, show that the ime of descent is

$$\sqrt{\left(\frac{6a}{g}\right)}.$$

34. In the year 1670, a scheme was proposed by Francis Lana or navigating the atmosphere by means of four large copper balls rom which the air had been exhausted. Each ball was to be 25 eet in diameter, and the metal to weigh 365 lbs.; determine the scending power.

The ascending power of each ball is 261 lbs.

35. A silk balloon, 40 feet in diameter, and weighing 80 lbs., is lled with hydrogen gas of specific gravity ·0694 (sp. gr. of air = 1); etermine the weight required to hold the balloon down.

The weight required is 2306 lbs.

36. A substance weighs w in a liquid whose specific gravity is

s_1, and w_2 in a liquid whose specific gravity is s_2; determine the specific gravity and weight of the substance.

The required sp. gr. is equal to $\frac{w_1 s_1 - w_2 s_2}{w_1 - w_2}$, and the weight is equal to $\frac{w_1 s_1 - w_2 s_2}{s_2 - s_1}$.

37. If 30 oz. of cedar, sp. gr. ·6, will just float in a liquid 5 lbs. of brass sp. gr. 8, what is the sp. gr. of the liquid?

Ans. 1·83.

38. If AB, AC, two sides of the triangle ABC, represent the directions of two forces whose magnitudes are represented by AB and 2AC respectively, then if the direction of the resultant cross BC in D, CD is ⅓BC, and the magnitude of the resultant is 3AD.

39. If AB, AC, AD, three conterminous edges of a triangular pyramid, represent three forces in magnitude and direction, the resultant will pass through E, the centre of gravity of the face BCD, and will be equal to 3AE.

40. If G be the centre of gravity of the triangular pyramid ABCD, show that the four forces represented by the lines GA, GB, GC, GD will be in equilibrium.

41. Show generally, from the equations of equilibrium, that if G be the centre of gravity of any number of equal weights placed at any points A, B, C, &c., then will the forces represented by GA, GB, GC, &c., be in equilibrium.

42. If G be the centre of gravity of n, equal weights placed at any points A, B, C, &c., and if O be any other point, then will the resultant of the n forces represented by OA, OB, OC, &c., be equal to n . OG.

43. Four equal and similar balls are placed in a row 5 feet apart on a smooth horizontal plane, the first is impelled towards the second with a velocity of 10 feet per second, after what interval will the fourth ball be struck, the modulus of elasticity being ½?

Ans. $2\frac{1}{18}$ sec.

Part III.

STATICS.

CHAPTER I.

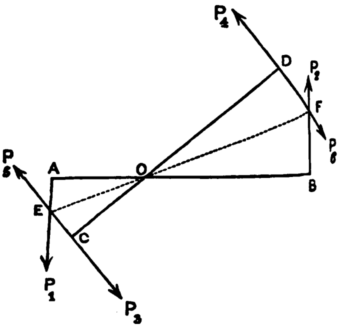
ON THE THEORY OF COUPLES.

335. Definitions. It has been seen that two equal non-con-urrent parallel forces do not admit of a single resultant, but form hat is termed a statical *couple*, having a tendency to produce a tatory motion. The perpendicular distance between the directions f the two forces is termed the *arm* of the couple; the product of ther force and the arm is termed the *moment* of the couple. et the direction in which the hands of a watch move be taken as e positive direction of rotation, then if a couple tend to produce tation in this direction, it is termed a *positive couple;* if in the ntrary direction, a *negative couple.* The direction in which any tatory motion appears to take place depends upon the position the observer. Every couple is therefore positive or negative, cording as it is viewed from the one side of its plane or from e other side. If it appear as a positive couple, that face of its ane upon which the observer is looking is termed its *positive ce;* and if it appear as a negative couple, the face is termed the *gative face.*

A line perpendicular to the plane of a couple, in length repre-nting the moment, is termed the *axis* of the couple. The axis positive when pointing away from the positive face, and negative en pointing away from the negative face.

336. *A couple may be turned in its own plane through any angle about any point in its arm, without altering its statical effect.*

Let AB be the arm of the couple, P_1 and P_2 the forces. Let O be any point in AB, and let the arm revolve about O into any new position CD. At C introduce two equal and opposite forces P_3 and P_5 acting perpendicularly to CD, and each equal to P_1 or P_2. Similarly, at D introduce P_4 and P_6. The line EO bisects the angle AEC, and FO the angle BFD, also EO and FO are in the same straight line.

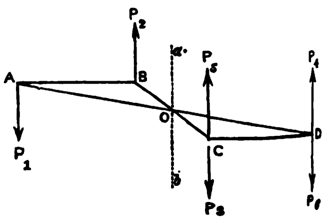


P_1 and P_5 are equivalent to some force in EO,
P_2 and P_6 „ an equal force in FO;

therefore P_1, P_2, P_5, P_6 are in equilibrium, and may be removed. Removing them, there will remain the forces P_3, P_4 acting in the arm CD. Hence, the couple P_3, CD, P_4 produces the same statical effect as the couple P_1, AB, P_2.

337. *A couple may be moved into any position parallel to itself, either in the same or in a parallel plane, without altering its statical effect.*

Let P_1, AB, P_2 be the original couple. Let CD be any line equal and parallel to AB. At C introduce two equal and opposite forces P_3 and P_5, each parallel and equal to P_1. Similarly, at D introduce P_4 and P_6.



Join A, D, and B, C. These lines will bisect each other at O. P_1 and P_6 are equivalent to $2P_1$ acting in the direction O*b*. P and P_5 are equivalent to $2P_1$

acting in the direction Oa. Hence, P_1, P_2, P_5, P_6 are in equilibrium, and may be removed. Removing them, there remains only the couple P_3, CD, P_4, which consequently produces the same statical effect as the original couple.

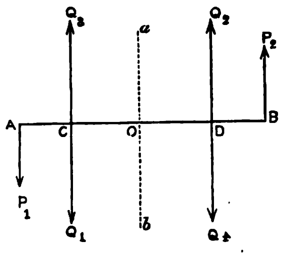
338. By combining the two preceding sections, we see that a couple may be removed to any position in its own or a parallel plane, without altering its statical effect; for, by the former section, it may be turned round until its arm be parallel to the new position, and then, by the latter section, it may be directly removed into that position.

339. *Two couples, in the same or parallel planes, and tending to produce rotation in the same direction, are equivalent, if their moments are equal.*

Let P_1, AB, P_2 be any couple. Bisect AB in O. Take any points C, D, equidistant from O. At C and D introduce two pairs of equal and opposite forces, Q_1, Q_3, and Q_2, Q_4; and let Q_1, Q_2, Q_3, Q_4 be of such magnitude that each, when multiplied by CD, $= P_1 \times AB$.

Since $Q_3 \times CD = P_2 \times AB$, it follows that $Q_3 \times CO = P_2 \times BO$.

Therefore, P_2 and Q_3 are equivalent to $P_2 + Q_3$, acting in the direction Oa.



Similarly, P_1 and Q_4 are equivalent to $P_1 + Q_4$, acting in the direction Ob.

Therefore the forces P_1, P_2, Q_3, Q_4 are in equilibrium, and may be removed. Removing them, there will remain only the couple Q_1 CD, Q_2 tending to produce rotation in the same direction as the original couple, and (since $Q_1 \times CD = P_1 \times AB$) having an equal moment.

340. *To find the resultant of any number of couples acting in the same or parallel planes.*

By Art. 339, the couples may be all replaced by others having equal arms; and, by Art. 338, they may then be removed until their arms coincide. At each extremity of the common arm there will then be a corresponding set of forces acting in the same straight line, and whose resultant is consequently their algebraic sum.

Let P, Q, R, &c., be the forces, and a, b, c, &c., their arms. Resolve these into couples having a common arm m, and let P′, Q′, R′, &c., be the reduced forces; then the resultant of all the couples will be a couple whose moment is

$$(P' + Q' + R' + \&c.)\, m\,;$$

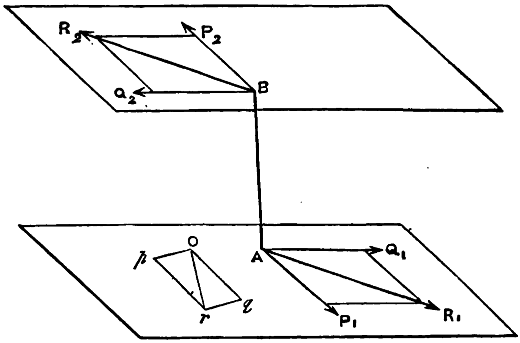
but $P'm = Pa$ and $Q'm = Qb$, and so on. Therefore,

$$(P' + Q' + R' + \&c.)\, m = Pa + Qb + Rc + \&c.\,;$$

whence the resultant is a couple whose moment equals the algebraic sum of the moments of the component couples.

341. *To find the resultant of two couples not acting in the same or parallel planes.*

If the arms of the given couples be unequal, replace them by others having equal arms. Let the planes of the couples intersect in the line AB, and let this be the common arm of the couples,



and let the reduced couples be P_1, AB, P_2, and Q_1, AB, Q_2. The planes of P_1, Q_1, and P_2, Q_2, are each perpendicular to AB, and are therefore parallel.

Let R_1 be the resultant of P_1 and Q_1, and R_2 the resultant of P_2 and Q_2; then R_1 and R_2 are equal and parallel, and consequently the resultant of the given couples is a couple R_1, AB, R_2.

Through any point O in the plane of P_1, Q_1, draw the lines Op and Oq at right angles to the directions of P_1 and Q_1, and both pointing away from the same face of the planes of the component couples. Make Op and Oq proportional respectively to P_1 and Q_1; then will the diagonal Or be perpendicular and proportional to R, and point away from the same face of the resultant couple. But since Op, Oq, and Or are proportional to P_1, Q_1, and R_1, they are also proportional to $P_1 \times AB$, $Q_1 \times AB$, and $R_1 \times AB$, that is, they are proportional to the moments of the several couples; therefore, Op, Oq, and Or are the axes of the three couples. Hence, *if two sides of a parallelogram meeting in any point represent the axes of two couples, both being positive or both negative axes, the diagonal drawn through the same point represents the corresponding axis of the resultant couple.*

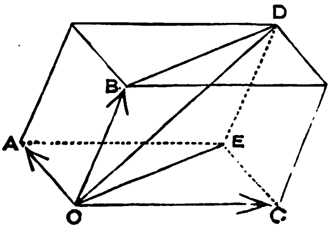
Let θ be the angle between P and Q, then the angle $pOq = \theta$. Hence, if L and M be the moments of the component couples, and G the moment of the resultant couple,

$$G_2 = L_2 + M_2 + 2\ LM \cos \theta.$$

342. *To find the magnitude and position of the resultant of three couples acting in planes at right angles to each other.*

Since the planes of the couples are at right angles to each other, heir axes will also be at right angles to each other.

Let AO, BO, and CO represent the axes of the couples, then, y the preceding Article, EO ill represent the axis of the ouple which is the resultant f the couples whose axes are O and CO; and, in like maner, DO will be the axis of the esultant of this and the third ouple, and, therefore, of the



EXAMPLES.

1. If three forces act upon a rigid body along the sides of a triangle, and are also represented in magnitude and direction by the sides of the triangle taken in order, the resultant is a couple whose moment is equal to twice the area of the triangle.

2. If forces act upon a rigid body along the sides of any rectilinear figure, and are also represented in magnitude and direction by the sides of the figure taken in order, the resultant is a couple whose moment equals twice the area of the figure.

3. If four forces act upon a rigid body along the sides of the figure formed by joining the extremities of two intersecting lines, and be represented in magnitude and direction by the sides of the figure taken in order, show that the resultant is a couple whose moment is equal to twice the difference of the vertically opposed triangles.

4. A beam of uniform thickness and density, whose weight is W and length $2a$, rests with one extremity A upon a smooth horizontal plane, and in contact with two pegs, one at B above the beam, and another at C below it; find the pressures on the plane and pegs.

Let b be the distance between the pegs, and α the inclination of the beam to the plane; then the pressure on the plane is equal to W, and the pressure on each of the pegs is equal to $\frac{Wa \cos \alpha}{b}$.

5. A beam leaning against a fixed peg, and with one extremity A upon a smooth horizontal plane, is kept at rest by a power P acting at A, at right angles with the beam; required the value of P when the beam is inclined to the plane at a given angle α.

Let h be the height of the pivot above the horizontal plane, W the weight of the beam, and a the distance of the centre of gravity of the beam from A; then the required distance is equal to

$$\frac{Wa \sin 2\alpha}{2h}.$$

6. The lower extremity of a weightless beam moves freely about fixed pivot, a weight W is suspended from the other extremity, nd the beam is supported in a given position by a cord attached t its centre at right angles to the beam; required the tension in he cord and the pressure upon the pivot.

Resolve the resistance of the pivot into two components acting rallel respectively to the tension and the weight, and apply the eorem of Art. 343.

Let α be the inclination of the beam to the horizontal line, then e tension in the cord is equal to $2W \cos \alpha$, and the pressure on e pivot is equal to W.

7. A beam, of uniform thickness and density, rests with one tremity upon a smooth horizontal plane, and the other upon a ooth vertical plane; the inclination of the plane is 60°; deterine the horizontal pressure which must be applied at the lower d of the beam that the inclination of the beam may be 30°; so find the pressures on the planes.

Let W be the weight of the beam, then the horizontal force quired is equal to $\frac{1}{4}W\sqrt{3}$; the pressure upon the inclined plane $\frac{1}{2}W$; and the pressure upon the horizontal plane is $\frac{3}{4}W$.

349. *To find the value of the resultant in terms of the components, and the angles comprised between their directions.*

In the expression $R^2 = (\Sigma . P\cos\alpha)^2 + (\Sigma . P\cos\beta)^2 + (\Sigma . P\cos\gamma)^2$ substitute for $\Sigma . P\cos\alpha$, $\Sigma . P\cos\beta$, and $\Sigma . P\cos\gamma$, their values; and since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have

$$\begin{aligned} R^2 = {} & P_1^2 + P_2^2 + P_3^2 + \&c. \\ & + 2P_1P_2(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2) \\ & + 2P_1P_3(\cos\alpha_1\cos\alpha_3 + \cos\beta_1\cos\beta_3 + \cos\gamma_1\cos\gamma_3) + \&c. \\ & + 2P_2P_3(\cos\alpha_2\cos\alpha_3 + \cos\beta_2\cos\beta_3 + \cos\gamma_2\cos\gamma_3) + \&c. \end{aligned}$$

But if $\widehat{P_1P_2}$ denote the angle between the directions of P_1 and P_2,

$$\cos\widehat{P_1P_2} = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$$

Hence,

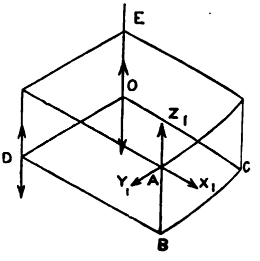
$$\begin{aligned} R^2 = {} & P_1^2 + P_2^2 + P_3^2 + \&c. \\ & + 2P_1P_2\cos\widehat{P_1P_2} + 2P_1P_3\cos\widehat{P_1P_3} + \&c. \\ & + 2P_2P_3\cos\widehat{P_2P_3} + 2P_2P_4\cos\widehat{P_2P_4} + \&c. \\ & + \&c. \end{aligned}$$

350. When forces act at different points of a rigid body they are not, as with forces acting at a single point, always reducible to a single force. Under a certain condition only, to be presently deduced, is this possible. They can, however, be always reduced to a single force and a single couple.

351. *To find the resultant force and the resultant couple, when any number of forces act at various points of a rigid body.*

Let P_1, P_2, &c., be the forces. Let $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, &c., be the angles which the forces make severally with any three rectangular axes, and $x_1, y_1, z_1, x_2, y_2, z_2$, &c., the co-ordinates of their several points of application.

Resolve P_1 into three forces acting in directions parallel to the axes; these will be $P_1\cos\alpha_1$, $P_1\cos\beta_1$, and $P_1\cos\gamma_1$.



Let A be the point of application of P_1, and X_1, Y_1, Z_1 the ree components into which P_1 has been resolved.

Let OC, OD, OE be the axes of x, y, z respectively, and OC, B, BA the co-ordinates of the point A; whence $OC = x_1$, $CB =$ $BA = z_1$. Through B draw BD parallel to OC; then $BD = x_1$, d $DO = y_1$.

At D and at O introduce two pairs of equal and opposite forces h equal to Z_1, and remove the point of application of Z_1 from to B. Then, instead of Z_1 at A, we have a force Z_1 acting along e axis of z; a positive couple, whose moment is $Z_1 \times DO$, or $Z_1 y_1$, ing in the plane of yz; and a negative couple, whose moment is $\times DB$, or $Z_1 x_1$, acting in a plane parallel to xz. Let the latter ple be removed into the plane of xz, and we have, instead of at A,

a force Z_1 acting along the axis of z,
a couple $Z_1 y_1$ acting in the plane of yz,
and a couple $-Z_1 x_1$ „ „ zx.

like manner, Y_1 at A may be resolved into

a force Y_1 acting along the axis of y,
a couple $Y_1 x_1$ acting in the plane of xy,
and a couple $-Y_1 z_1$ „ „ yz.

d X_1 at A may be resolved into

a force X_1 acting along the axis of x,
a couple $X_1 z_1$ acting in the plane of zx,
and a couple $-X_1 y_1$ „ „ xy.

Whence, by adding together the moments of those couples which in the same plane (Art. 340), the force P_1 is resolved into the ces X_1, Y_1, Z_1 along the axes and into the couples

$$Z_1 y_1 - Y_1 z_1 \text{ in plane of } yz,$$
$$X_1 z_1 - Z_1 x_1 \quad „ \quad zx,$$
$$Y_1 x_1 - X_1 y_1 \quad „ \quad xy.$$

solving the other forces P_2, P_3, &c., in a similar manner, we ain, in place of the original forces, the forces

$\Sigma . X$ acting along the axis of x,
$\Sigma . Y$ „ „ y,
$\Sigma . Z$ „ „ z;

and the couples $\Sigma(Zy - Yz)$ in plane of yz,
$\Sigma(Xz - Zx)$ „ zx,
$\Sigma(Yx - Xy)$ „ xy.

Let L, M, N represent severally these couples; then, if R be the resultant force, and G the moment of the resultant couple, by Articles 342, 345,

$$R^2 = (\Sigma . X)^2 + (\Sigma . Y)^2 + (\Sigma . Z)^2,$$

and $$G^2 = L^2 + M^2 + N^2.$$

And if α, β, γ be the angles which the direction of R makes with the axes, and λ, μ, ν those which the axis of G makes with the axes,

$$\cos\alpha = \frac{\Sigma . X}{R},\ \cos\beta = \frac{\Sigma . Y}{R},\ \cos\gamma = \frac{\Sigma . Z}{R},$$

$$\text{and } \cos\lambda = \frac{L}{G},\quad \cos\mu = \frac{M}{G},\quad \cos\nu = \frac{N}{G}.$$

352. *To find the resultant of any number of forces acting at various points of a rigid body, when all the forces act in the same plane.*

Let the plane of the forces be the plane of xy, then γ_1, γ_2 &c., will each $= 90^\circ$, and z_1, z_2, &c., will each $= 0$. Also, $\alpha_1 + \beta_1 = 90^\circ$, $\alpha_2 + \beta_2 = 90^\circ$, &c.

Hence the given forces, resolved as in the preceding Article, will be reduced to forces $\Sigma . X$ and $\Sigma . Y$ acting along the axes of x and y respectively, and a couple whose moment is G, the values of $\Sigma . X$, $\Sigma . Y$, and G, being those given by the following equations:—

$$\Sigma . X = \Sigma . P\cos\alpha,$$
$$\Sigma . Y = \Sigma . P\sin\alpha,$$
$$G = \Sigma(Yx - Xy).$$

If $\Sigma . X$ and $\Sigma . Y$ each vanish, the resultant of all the forces will be the couple whose moment is G. If $\Sigma . X$ and $\Sigma . Y$ do not both vanish, their resultant and the resultant couple may be combined into a single force, which will consequently be the resultant of the given forces. Let R be this resultant, α the angle its direction makes with the axis of x, and x, y the co-ordinates of any point in its direction. Resolve R into forces acting along the axes and a couple, these must be equal to $\Sigma . X$, $\Sigma . Y$, and G respectively. Hence,

$$R \cos a = \Sigma . X, \qquad R \sin a = \Sigma . Y,$$

and $$R(x \sin a - y \cos a) = G.$$

Whence $$R^2 = (\Sigma . X)^2 + (\Sigma . Y)^2,$$

and $$x(\Sigma . Y) - y(\Sigma . X) = G.$$

The former of these equations determines the magnitude of R, and the latter the line of its direction.

353. *When any number of forces act at various points of a rigid body, to find the condition that they may have a single resultant.*

It has already been shown, that when any number of forces act at various points of a rigid body, the resultant force $= \sqrt{\{(\Sigma . X)_2 + (\Sigma . Y)^2 + (\Sigma . Z)^2\}}$, and the resultant couple $= \sqrt{(L^2 + M^2 + N^2)}$.

If $\Sigma . X = 0$, $\Sigma . Y = 0$, $\Sigma . Z = 0$, there is no resultant force, but resultant couple only; hence, in this case, the given forces do not admit of a single resultant.

If $L = 0$, $M = 0$, and $N = 0$, there is no resultant couple, and ence the given forces will be reduced to a single resultant passing hrough the origin.

Let neither $\Sigma . X$, $\Sigma . Y$, and $\Sigma . Z$, nor L, M, and N, vanish ogether, and let R be the resultant, a, b, c the angles which its irection makes with the axes, and x, y, z the co-ordinates of a oint in its direction. Let R be resolved, in the same manner as he forces $P_{,,}$ &c. (Art. 351), into three forces acting along the xes, and into three couples acting in the co-ordinate planes; these must be equal severally to $\Sigma . X$, $\Sigma . Y$, $\Sigma . Z$, L, M, N. Hence,

$$R \cos a = \Sigma . X, \qquad R \cos b = \Sigma . Y, \qquad R \cos c = \Sigma . Z.$$

$$R(y \cos c - z \cos b) = L,$$
$$R(z \cos a - x \cos c) = M,$$
$$R(x \cos b - y \cos a) = N.$$

Substituting, in the last three equations, the values of R cos a, &c., given by the first three, we obtain

$$y(\Sigma . Z) - z(\Sigma . Y) = L,$$
$$z(\Sigma . X) - x(\Sigma . Z) = M,$$
$$x(\Sigma . Y) - y(\Sigma . X) = N.$$

Multiplying both sides of the first equation by $\Sigma . X$, of the second by $\Sigma . Y$, and of the third by $\Sigma . Z$, and adding together, x, y, and z will be eliminated. The resulting equation will be

$$L(\Sigma . X) + M(\Sigma . Y) + N(\Sigma . Z) = 0,$$

which is the condition required.

354. *To find the conditions of equilibrium, when any number of forces act at various points of a rigid body.*

In order that there may be equilibrium, the resultant force and the resultant couple must both vanish. But that R may vanish, $\Sigma . X$, $\Sigma . Y$, $\Sigma . Z$ must each vanish; and that G may vanish, L, M, and N must each vanish. Hence, the six conditions of equilibrium are

$$\Sigma . P \cos \alpha = 0,$$
$$\Sigma . P \cos \beta = 0,$$
$$\Sigma . P \cos \gamma = 0.$$
$$\Sigma . P(y \cos \gamma - z \cos \beta) = 0,$$
$$\Sigma . P(z \cos \alpha - x \cos \gamma) = 0,$$
$$\Sigma . P(x \cos \beta - y \cos \alpha) = 0.$$

In like manner, if all the forces act in one plane, the conditions of equilibrium are three only; viz.,

$$\Sigma . P \cos \alpha = 0,$$
$$\Sigma . P \sin \alpha = 0,$$
$$\Sigma . P(x \sin \alpha - y \cos \alpha) = 0.$$

355. *To find the conditions of equilibrium of forces acting at various points of a rigid body, when one point in the body is fixed.*

Let the fixed point be taken as the origin. There will evidently be equilibrium, if all the forces have a single resultant passing through the origin. Hence, by Art. 353, the conditions required are that

$$L = 0, \qquad M = 0, \qquad N = 0.$$

If all the forces act in one plane, the only condition (Art. 352) is that $G = 0$, or that $\Sigma . P(x \sin \alpha - y \cos \alpha) = 0$.

But if p_1 be the length of the perpendicular drawn from the origin to the line of direction of P_1,

$$p_1 = x_1 \sin \alpha_1 - y_1 \cos \alpha_1;$$

and therefore the condition of equilibrium may be written,

$$\Sigma . Pp = 0.$$

356. *To find the conditions of equilibrium of forces acting at various points of a rigid body, when two points in the body are fixed.*

Let the line joining the given points be taken as the axis of z; then, since the body is fixed upon this axis, the forces acting along this line cannot produce motion, nor can the forces acting along the axes of x any y, since they tend only to press the body against the fixed axis; nor can the couples in the planes of yz and zx produce any motion. Hence, the only condition of equilibrium is that

$$\Sigma \,.\, P(x \cos \beta - y \cos \alpha) = 0.$$

If the body were free to slide along the axis of z, then, in addition to the preceding condition, the forces acting along the axis of z must also vanish, or

$$\Sigma \,.\, P \cos \gamma = 0.$$

EXAMPLES.

1. If three forces acting upon a rigid body at any points be in equilibrium, show from the general equations that they must lie n the same plane.

2. Show that any number of forces, not in equilibrium, acting n different directions, at various points of a rigid body, may be always reduced to *two* single forces.

3. If R be the resultant of the forces P_1, P_2, &c., acting in one plane, at various points of a rigid body, show that

$$\cos \widehat{RP_1} = \frac{P_1 \cos \widehat{P_1P_1} + P_2 \cos \widehat{P_1P_2} + P_3 \cos \widehat{P_1P_3} + \&c.}{R},$$

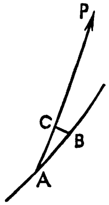
the angle $\widehat{P_1P_1}$ (which, of course, $= 0$) being introduced for the sake of symmetry.

4. Show that the preceding expression holds good, if sines be written for cosines.

CHAPTER III.

ON VIRTUAL VELOCITIES.

357. Let A, the point of application of a force P, be displaced from A to B. Draw BC perpendicular to the direction of P, then the distance AC is termed the *virtual velocity* of the point A with respect to the force P.



The virtual velocity is taken to be positive when, as in the figure, the perpendicular falls on the same side of the point of application as the point *towards* which the force acts. If the perpendicular falls on the opposite, the virtual velocity is reckoned negative. If the direction of P make an angle α with the line AB, the virtual velocity is equal to

$$AB \cos \alpha.$$

358. *If any number of forces acting upon a point be in equilibrium, and the point of application be displaced, the sum of the products of each force into its virtual velocity is equal to zero.*

Let P_1, P_2, &c., be the given forces, and v_1, v_2, &c., their virtual velocities; then shall $P_1 v_1 + P_2 v_2 + \&c. = 0$.

Let α_1, α_2, &c., be the angles which the directions of the forces make with the line joining the two positions of the point of application. Then, if this line be taken as the axis of x, the conditions of equilibrium give

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0.$$

Let a be the distance between the two positions of the given point, then $P_1 a \cos \alpha_1 + P_2 a \cos \alpha_2 + \&c. = 0$.

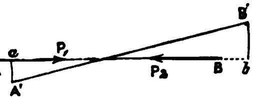
But (Art. 357) $v_1, = a \cos \alpha_1$, $v_2 = a \cos \alpha_2$, &c.

$$P_1 v_1 + P_2 v_2 + \&c. = 0;$$

or, employing the notation explained in the preceding chapter,

$$\Sigma . Pv = 0.$$

359. *If two equal and opposite forces act at two points in a rigid body, and these points be displaced through an indefinitely small space, the virtual velocities of the forces shall be equal in magnitude, and of opposite signs.*



Let P_1, P_2, be equal and opposite forces acting at the points A and B in a rigid body. Let the line AB be supposed to lie in the plane of the paper. Let A be taken as the origin f co-ordinates, and the line AB as the xis of x; and let the distance AB = c.

Let A and B be displaced through indefinitely small spaces to he points A′ and B′, situated either within or without the plane of he paper. Let x_1, y_1, z_1, be the co-ordinates of A′, and $c+x_2$, y_2, „ those of B′; x_1, y_1, z_1, x_2, y_2, z_2, being indefinitely small quanities. Then $x_1 = \text{A}a$, or the virtual velocity of P_1 and $x_2 = \text{B}b$, or he virtual velocity of P_2.

Since the points A, B, are rigidly connected, A′B′ = AB = c;

$$\therefore \quad c^2 = (c + x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.\text{*}$$

But since x_1, y_1, z_1, x_2, y_2, z_2, are indefinitely small quantities, heir squares and the products of any two of them may be disrearded. Rejecting all such terms, the equation becomes

$$c_2 = c_2 + 2cx_2 - 2cx_1;$$

$$\therefore \quad 0 = 2cx_2 - 2cx_1,$$

$$\text{or,} \quad x_1 = x_2.$$

ence the virtual velocities of P_1 and P_2 are equal in magnitude. lso, x_1 is measured along the direction of P_1, and is therefore ositive; but x_2 is not measured along the direction of P_2, and is erefore negative; the virtual velocities, therefore, have contrary gns.

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* If a_1, b_1, c_1, a_2, b_2, c_2, be the co-ordinates of two points, the square of e distance between them $= (a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2$. Let A, fig. rt. 351, be the point whose co-ordinates are a_1, b_1, c_1, and O the point hose co-ordinates are a_2, b_2, c_2. The distance AO is the diagonal of a right lid whose sides are severally $a_1 - a_2$, $b_1 - b_2$, $c_1 - c_2$, and therefore

$$\text{AO}^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2.$$

360. *If any number of forces acting at various points of a rigid body be in equilibrium, and if the body receive an indefinitely small disturbance, the sum of the products of each force into its virtual velocity is equal to zero.*

Let P_1, P_2, &c., be the forces, A_1, A_2, &c., their points of application, and v_1, v_2, &c., their virtual velocities; it is required to show that $P_1v_1 + P_2v_2 + \&c. = 0$.

Since A_1, A_2, &c. are points in a rigid body, we may suppose them connected by rigid rods. Let $Q_{2 \cdot 1}$ denote the force exerted on A_1, from its connection with A_2, and $Q_{1 \cdot 2}$ the force exerted on A_2, from its connection with A_1. These are evidently equal forces, and act along the line AA_2. In like manner, let $Q_{3 \cdot 1}$ $Q_{1 \cdot 3}$ denote the forces acting along A_1A_3, $Q_{2 \cdot 3}$ $Q_{3 \cdot 2}$ those along A_2A_3, and so on.

If, then, instead of the rods, we suppose the forces $Q_{1 \cdot 2}$, $Q_{1 \cdot 3}$, $Q_{2 \cdot 3}$, &c., to act at the various points of the given body, the forces acting at each point must severally be in equilibrium.

Let $v_{1 \cdot 2}$ be the virtual velocity of $Q_{1 \cdot 2}$, $v_{2 \cdot 3}$, that of $Q_{2 \cdot 3}$, and so on. Then, by Art. 283,

$$\begin{gathered} P_1v_1 + Q_{2 \cdot 1}\, v_{2 \cdot 1} + Q_{3 \cdot 1}\, v_{3 \cdot 1} + \&c. = 0, \\ P_2v_2 + Q_{1 \cdot 2}\, v_{1 \cdot 2} + Q_{3 \cdot 2}\, v_{3 \cdot 2} + \&c. = 0, \\ P_3v_3 + Q_{1 \cdot 3}\, v_{1 \cdot 3} + Q_{2 \cdot 3}\, v_{2 \cdot 3} + \&c. = 0. \\ \&c. \qquad\qquad \&c. \end{gathered}$$

But since A_1, A_2, &c., have been displaced through an indefinitely small space, it follows from the preceding proposition that $v_{2 \cdot 1} = -v_{1 \cdot 2}$, $v_{2 \cdot 3} = -v_{3 \cdot 2}$, &c. Hence, in the foregoing series of equations, for each term arising from the mutual action of the points of the body there will be a corresponding term, equal in magnitude but of contrary sign. Therefore, if the several equations be added together, these terms will mutually destroy each other. Whence, $P_1v_1 + P_2v_2 + P_3v_3 + \&c. = 0$.

361. It will be seen, from Art. 357, that when the line joining the two positions of the point of application forms a right angle with the direction of the force, the virtual velocity is equal to

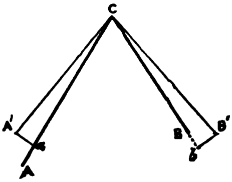
zero. Hence, the virtual velocities of all resistances, when bodies rest upon a perfectly smooth immoveable plane, are severally equal to zero. Also, the virtual velocities of all reactions at fixed points must, it is plain, be severally equal to zero. Hence, when forces acting upon a rigid body are in equilibrium, some of which are resistances of smooth planes and reactions of fixed points, these latter may be disregarded, and the principle of virtual velocities holds good of the remaining forces.

362. *If two material particles, in equilibrium under any forces, be connected by an inextensible cord, the virtual velocities of the tensions in the cord are equal in magnitude, but have opposite signs.*

This is evidently true when the displacement takes place in the direction of the cord itself.

If the displacement do not take place in the direction of the cord:—first, let the cord form a single straight line; then, since the particles preserve an invariable distance, the cord being supposed inextensible, the proposition is identical with that already proved in Art. 359.

Secondly, let the cord pass round a fixed point so as to form two straight lines. Let ACB be the original position of the cord. Let A and B be displaced over an indefinitely small space to A′ and B′, the positions of A′ and B′ being such as to allow the cord to remain stretched. Then A′C + CB′ = AC + CB. With C as the centre, and CA′, CB′ as radii, describe the small arcs $A'a$, $B'b$. Since the displacement is indefinitely small, these arcs may be regarded as straight lines perpendicular to AC and BC, and therefore Aa and Bb are the virtual velocities of A and B. But $AC = A'C + Aa$, and $BC = B'C - Bb$;



$$\therefore \qquad AC + BC = A'C + B'C + Aa - Bb;$$

$$\therefore \qquad 0 = Aa - Bb,$$

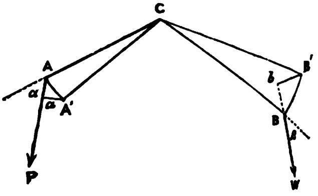
$$\text{or} \qquad Aa = Bb.$$

Hence, the virtual velocities are equal in magnitude. The tension on A acts towards C, therefore Aa falling along the direction is positive. The tension on B acts towards C, and therefore Bb is negative. Hence, the virtual velocities have contrary signs.

363. From the preceding Articles, it follows that if a system of rigid bodies, connected by flexible inextensible cords, be in equilibrium under any forces, the principle of virtual velocities applies to the forces irrespective of the tensions in the cords, since the products arising from these enter in pairs which mutually destroy each other, and (Art. 361) irrespective also of the resistance of smooth planes and the reactions of fixed points.

364. *To verify the principle of virtual velocities in the bent lever.*

Let ACB be the lever, and C its fulcrum. Let the forces P and W make the angles α and β respectively with the arms produced. Then the condition of equilibrium is (Art. 72) that $P \, . \, AC \sin\alpha = W \, . \, BC \sin\beta$. Let A′CB′ be the position of the lever after displacement. Since the displacement is indefinitely small, AA′ and BB′ may be regarded as straight lines perpendicular to AC and BC respectively, and therefore the angle $A'Aa = 90^\circ - \alpha$, and the angle $B'Bb = 90^\circ - \beta$. By the principle of virtual velocities



$$P \, . \, Aa - W \, . \, Bb = 0.$$

But $\qquad Aa = AA' \cos A'Aa = AA' \sin\alpha,$

and $\qquad Bb = BB' \cos B'Bb = BB' \sin\beta;$

$$\therefore \qquad P \, . \, AA' \sin\alpha = W \, . \, BB' \sin\beta,$$

or $\qquad P \sin\alpha : W \sin\beta :: BB' : AA';$

but since the triangles ACA′, BCB′ are similar,

$$BC : AC :: BB' : AA',$$

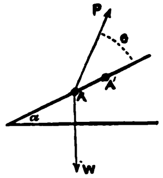
$$\therefore \qquad P \sin\alpha : W \sin\beta :: BC : AC,$$

which is identical with the condition of equilibrium stated above.

365. *To verify the principle of virtual velocities in the inclined plane.*

Let a weight W be sustained on an inclined plane at the point A by a power P, whose direction makes an angle θ with the plane. Let α be the inclination of the plane. Then, by Art. 100,

$$P : W :: \sin\alpha : \cos\theta.$$



Let A be displaced to A′ then since the irection of P makes an angle θ with AA′ Art. 357),

$$\text{P's virtual velocity} = AA' \cos\theta;$$

nd since the direction of W makes an angle $90^\circ - \alpha$ with AA′,

$$\text{W's virtual velocity} = AA' \cos(90^\circ - \alpha) = AA' \sin\alpha,$$

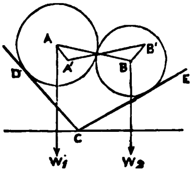
nd not falling along the direction of the force is negative. Therere, by the principle of virtual velocities,

$$P . AA' \cos\theta - W . AA' \sin\alpha = 0;$$

$$\therefore \qquad P \cos\theta = W \sin\alpha,$$

e same relation as that given above.

366. *To apply the principle of virtual velocities to determine the osition of equilibrium of two spheres, of uniform density, resting pon two smooth inclined planes.*



Let A, B be the centres of two spheres whose weights are W_1, $_2$ respectively, resting on the smooth inined planes CD, CE, whose inclinations are , α_2. In the position of equilibrium, let θ e the angle which the line AB makes with e vertical line through A. Let the system ceive an indefinitely small displacement, in nsequence of which A is moved to A′, and to B′. The lines AA′, BB′ are parallel respectively to CD and E; therefore the angle between AA′ and the direction of $W_1 =$ $^\circ - \alpha_1$, and the angle between BB′ and the direction of $W_2 =$ $^\circ - \alpha_2$. Hence,

$$\text{virtual velocity of } W_1 = AA' \sin\alpha_1,$$

$$\text{,,} \qquad \text{,,} \qquad W_2 = BB' \sin\alpha_2;$$

the former is positive and the latter negative. Therefore, by the principle of virtual velocities,

$$W_1 \,.\, AA' \sin \alpha_1 - W_2 \,.\, BB' \sin \alpha_2 = 0. \qquad \text{(i.)}$$

But since the distance between A and B is invariable, the virtual velocities of A and B in the direction of AB are equal in magnitude and have contrary signs (Art. 359);

$$\therefore \qquad AA' \cos A'AB = -BB' \cos B'BA.$$

But the angle $A'AB = \theta - (90^\circ - \alpha_1) = \theta + \alpha_1 - 90^\circ$; and the angle $B'BA = \theta + 90^\circ - \alpha_2 = \theta - \alpha_2 + 90^\circ$;

$$\therefore \qquad AA' \sin (\theta + \alpha_1) = BB' \sin (\theta - \alpha_2). \qquad \text{(ii.)}$$

Eliminating AA′ and BB′ between the equations i. and ii. we obtain

$$W_1 \sin \alpha_1 \sin (\theta - \alpha_2) = W_2 \sin \alpha_2 \sin (\theta + \alpha_1).$$

Divide both sides of this equation by $\sin \alpha_1 \sin \alpha_2 \cos \theta$, then

$$W_1 (\tan \theta \cot \alpha_2 - 1) = W_2 (\tan \theta \cot \alpha_1 + 1);$$

$$\therefore \qquad \tan \theta = \frac{W_1 + W_2}{W_1 \cot \alpha_2 - W_2 \cot \alpha_1}.$$

367. The method pursued in the preceding will serve also to determine the position of equilibrium of a bar resting on two smooth inclined planes. For, let AB be the bar, and AA′, BB′ portions of the planes on which it is resting. The inclinations of these planes are α_1 and α_2 respectively. Let W be the weight of the bar, resolve W acting at the centre of gravity of the bar into two forces W_1 and W_2, acting at A and B respectively. Then, as in the preceding,

$$\tan \theta = \frac{W_1 + W_2}{W_1 \cot \alpha_2 - W_2 \cot \alpha_1}.$$

Let a and b be the distance of the centre of gravity of the bar from the points A and B respectively, then

$$W_1 = W \,.\, \frac{b}{a+b}, \text{ and } W_2 = W \frac{a}{a+b}.$$

Substituting these values in the expression just given, and

$$\tan \theta = \frac{a+b}{b \cot \alpha_2 - a \cot \alpha_1}.$$

Or, if ϕ be the angle which the beam, in the position of equilibrium, makes with the horizontal line, since $\phi = 90' - \theta$,

$$\tan \phi = \frac{b \cot \alpha_2 - a \cot \alpha_1}{a + b}.$$

368. *If any system of rigid bodies is in equilibrium under the action of no forces but their weights, tensions in inextensible cords, resistances of smooth immoveable planes, and the reactions of fixed points, and if an indefinitely small motion be communicated to its parts, the centre of gravity will neither ascend nor descend.*

Let P_1, P_2, &c., be the weights of the several particles of the system, and z_1, z_2, &c., the distances of their centres of gravity from any horizontal plane; then, if h be the distance of the centre of gravity from the same plane (Art. 64),

$$h = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \&c.}{P_1 + P_2 + P_3 + \&c.}.$$

Let the system receive an indefinitely small displacement, and let h, z_1, z_2, &c., become h', $z_1 + v_1$, $z_2 + v_2$, &c.; then

$$h' = \frac{P_1 (z_1 + v_1) + P_2 (z_2 + v_2) + P_3 (z_3 + v_3) + \&c.}{P_1 + P_2 + P_3 + \&c.};$$

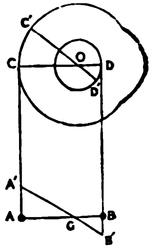
$$\therefore \qquad h' - h = \frac{P_1 v_1 + P_2 v_2 + P_3 v_3 + \&c.}{P_1 + P_2 + P_3 + \&c.}.$$

But, by Art. 363, the principle of virtual velocities is applicable to the external forces alone, irrespective of the tensions, resistances, &c. Therefore, $P_1 v_1 + P_2 v_2 + P_3 v_3 + \&c. = 0$;

$$\therefore \qquad h' - h = 0; \text{ or } h' = h.$$

369. *To verify the preceding theorem in the wheel and axle and the double inclined plane.*

First: Let P and W be in equilibrium in the wheel and axle. Let R be the radius of the wheel, and r the radius of the axle, and let A, B be the respective positions of P and W when in the same horizontal line. The centre of gravity of P and W is evidently in the line AB. Let the machine move on its axis through an angle COC′, in consequence of which A moves to A′, and B to B′. Then AA′ is equal to the arc CC′, and BB′ to the arc DD′. But CC′, DD′ are similar arcs of circles, and are therefore as the radii, or $CC' : DD' :: R : r$;



$\therefore$ $AA' : BB' :: R : r$.

But $P : W :: r : R$;

$\therefore$ $P : W :: BB' : AA'$.

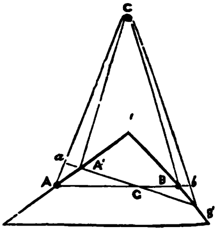
Also, since the triangles AGA′, BGB′ are similar,

$$BB' : AA' :: B'G : A'G;$$

$\therefore$ $P : W :: B'G : A'G$,

or G, a point in the horizontal line AB, is the centre of gravity of P and W in their new position.

Secondly: Let the weights P and W, connected by a cord passing over a fixed pulley at C, be in equilibrium upon a double inclined plane. Let a_1, a_2 be the inclinations of the planes, and β_1, β_2 the angles made by the cord with the planes. Let T be the tension in the cord; then, since T, P, and the resistance at A are in equilibrium, by Art. 100,



$$P : T :: \cos\beta_1 : \sin a_1.$$

Similarly, $W : T :: \cos\beta_2 : \sin a_2$;

$\therefore$ $P : W :: \cos\beta_1 \sin a_2 : \cos\beta_2 \sin a_1$.

Let A and B be the positions of P and W when in the same horizontal line. Let A and B be moved through indefinitely small spaces to A′ and B′. With C as a centre, describe the arcs A′*a*, B*b*. Then A*a* and B′*b*, being the portions by which the two parts of the cord are severally shortened and lengthened, are equal. The arcs A′*a*, B*b* being indefinitely small, may be regarded as straight lines perpendicular to AC and B′C respectively; that is, $Aa = AA' \cos\beta_1$, and $B'b = BB' \cos\beta_2$;

$\therefore$ $AA' \cos\beta_1 = BB' \cos\beta_2$.

Let the angle $AGA' = \theta$, then

$$A'G = AA' \frac{\sin \alpha_1}{\sin \theta},$$

and $$B'G = BB' \frac{\sin \alpha_2}{\sin \theta},$$

$$\therefore \quad \frac{A'G}{B'G} = \frac{AA' \sin \alpha_1}{BB' \sin \alpha_2} = \frac{\cos \beta_2 \sin \alpha_1}{\cos \beta_1 \sin \alpha_2}.$$

$$\therefore \quad A'G : B'G :: W : P,$$

or G is the centre of gravity of P at A′ and W at B′.

370. The importance of the principle of virtual velocities has led to attempts to supply a demonstration of it independently of the equations of equilibrium deduced in the preceding chapter, but none of these are sufficiently elementary in their character to be introduced here. If, however, this principle be assumed, the conditions of equilibrium may be deduced from it.

371. *To deduce the conditions of equilibrium of a rigid body from the principle of virtual velocities.*

Every possible motion of which any rigid body is susceptible may be resolved into a motion of translation and a motion of rotation round a fixed axis.

First, let the body receive a motion of translation, such that all the points in it move over a distance m in the direction of the line which makes with the axes angles whose cosines are a, b, c, respectively. Then, if x, y, z, be the co-ordinates of the point of application of any one of the forces P, and α, β, γ, the angles which the direction of P makes with the axes; and if v be the virtual velocity of P,

$$v = m\,(a \cos \alpha + b \cos \beta + c \cos \gamma).$$

By the principle of virtual velocities,

$$\Sigma \,.\, Pv = 0\,;$$

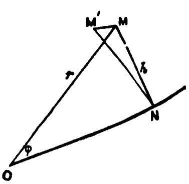
therefore,

$$a\,\Sigma \,.\, P \cos \alpha + b\,\Sigma \,.\, P \cos \beta + c\,\Sigma \,.\, P \cos \gamma = 0.$$

But this result is independent of the values of a, b, and c, for the direction of the axes is altogether arbitrary. Consequently, each term must separately vanish, or we must have

$$\left.\begin{aligned}\Sigma\,.\,\mathrm{P}\cos\alpha&=0,\\ \Sigma\,.\,\mathrm{P}\cos\beta&=0,\\ \Sigma\,.\,\mathrm{P}\cos\gamma&=0.\end{aligned}\right\}\quad\text{(i.)}$$

Again, let the body revolve through a small angle θ about a line passing through the origin, and making with any three rectangular axes angles whose cosines are a, b, c, respectively. Let ON represent this line. Let M be the point of application of any one of the forces, and x, y, z, its co-ordinates, r its distance from the origin, and h its perpendicular distance from ON. Let M revolve about the axis ON, through a small angle θ into the position M. The small arc MM′ may be regarded as a straight line perpendicular to the plane MON. Let the angles which MM′ makes with the axes have for their cosines A, B, C respectively, and let the force acting at M make with the axes the angles α, β, γ. Then, if v be the virtual velocity,



$$v=\mathrm{MM}'(\mathrm{A}\cos\alpha+\mathrm{B}\cos\beta+\mathrm{C}\cos\gamma).$$

The angles which OM makes with the axes have for their cosines $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, respectively. Hence, since MM′ is perpendicular to OM,

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}z=0.$$

Similarly, because the angle between MM′, and a line through M parallel to ON, is a right angle,

$$\mathrm{A}a+\mathrm{B}b+\mathrm{C}c=0.$$

From these two equations, it follows that

$$\mathrm{A}:\mathrm{B}:\mathrm{C}::bz-cy:cx-az:ay-bx.^{*}$$

---

* The student will find it useful to remember the following result, which he may easily deduce. Given two equations between three quantities, x, y, z, of the form

$$a_1x+b_1y+c_1z=0,$$
$$a_2x+b_2y+c_2z=0;$$

then $x:y:z::b_1c_2-b_2c_1:c_1a_2-c_2a_1:a_1b_2-a_2b_1$.

But $A^2 + B^2 + C^2 = 1$; whence, letting K stand as an abbreviation for

$$\sqrt{(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2},$$

we obtain

$$A = \frac{bz - cy}{K}, \qquad B = \frac{cx - az}{K}, \qquad C = \frac{ay - bx}{K}.$$

Again, $MM' = h\theta$. But if the angle $MON = \phi$,

$$h = r \sin \phi,$$

and

$$\cos \phi = \frac{ax + by + cz}{r};$$

whence, remembering that $a^2 + b^2 + c^2 = 1$, and that $x^2 + y^2 + z^2 = r^2$, t results that

$$h = \sqrt{(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2},$$

or

$$h = K.$$

Substituting the values of A, B, C, and MM′ just obtained in the xpression $v = MM'(A \cos \alpha + B \cos \beta + C \cos \gamma)$, we have

$$= \theta\{(bz - cy) \cos \alpha + (cx - az) \cos \beta + (ay - bx) \cos \gamma\},$$

$$= \theta\{a(y \cos \gamma - z \cos \beta) + b(z \cos \alpha - x \cos \gamma) + c(x \cos \beta - y \cos \alpha)\}.$$

By the principle of virtual velocities,

$$\Sigma . Pv = 0.$$

Therefore,

$$\begin{aligned} &a\, \Sigma . P(y \cos \gamma - z \cos \beta) \\ &+ b\, \Sigma . P(z \cos \alpha - x \cos \gamma) \\ &+ c\, \Sigma . P(x \cos \beta - y \cos \alpha) = 0. \end{aligned}$$

As before, this result is independent of the direction of the ces, and therefore of a, b, and c. Consequently, each term must anish separately; that is, we must have

$$\left.\begin{aligned} \Sigma . P(y \cos \gamma - z \cos \beta) = 0, \\ \Sigma . P(z \cos \alpha - x \cos \gamma) = 0, \\ \Sigma . P(x \cos \beta - y \cos \alpha) = 0, \end{aligned}\right\} \quad \text{(ii.)}$$

Equations i. and ii. correspond with the six equations of equibrium.

CHAPTER IV.

ON THE CENTRE OF GRAVITY.

372. It has been shown in the chapter on the centre of gravity in Part i., that the centre of gravity of any body can be readily found, when the centres of gravity of all its parts are known.

Thus, if M_1, M_2, &c., be the magnitudes of the several parts, and x_1, x_2, &c., the distances of their centres of gravity from any plane; then, if $\bar{x}$ be the distance of the centre of gravity from the same plane,

$$\bar{x} = \frac{M_1x_1 + M_2x_2 + \&c.}{M_1 + M_2 + \&c.},$$

or, according to the notation used in previous chapters,

$$\bar{x} = \frac{\Sigma . Mx}{\Sigma . M}.$$

If M_1, M_2, &c., be equal to one another, and n in number, then $\Sigma . Mx = M . \Sigma x$, and $\Sigma . M = nM$, therefore, in this case

$$\bar{x} = \frac{\Sigma . x}{n}.$$

373. Whenever a figure is symmetrical about any axis, then if of uniform density, the centre of gravity will lie in this axis, and may be found by finding its distance from any plane at right angles with the axis. When the figure is not symmetrical about any axis, the position of the centre of gravity is found by finding its distances from three planes at right angles with one another.

As before, let M_1, M_2, &c., be the magnitudes of the several parts; x_1, x_2, &c., the distances of their centres of gravity from the first plane; y_1, y_2, &c., their distances from the second plane; and z_1, z_2, &c., their distances from the third plane: then if $\bar{x}$, $\bar{y}$, the distances of the centre of gravity of the whole from planes,

$$\bar{x} = \frac{\Sigma . Mx}{\Sigma . M},$$

$$\bar{y} = \frac{\Sigma . My}{\Sigma . M},$$

$$\bar{z} = \frac{\Sigma . Mz}{\Sigma . M},$$

given figure be a plane, it is sufficient to find the distances
ntre of gravity from two lines intersecting at right angles,
position of the centre of gravity is given by the two

$$\bar{x} = \frac{\Sigma . Mx}{\Sigma . M},$$

$$\bar{y} = \frac{\Sigma . My}{\Sigma . M}.$$

If the centre of gravity be the origin of co-ordinates
$= 0$, and $\bar{z} = 0$; and therefore in this case,

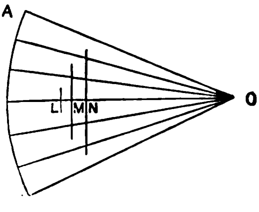
$$\Sigma . Mx = 0,$$

$$\Sigma . My = 0,$$

$$\Sigma . Mz = 0.$$

To find the centre of gravity of the sector of a circle.

be the radius of the circle, and
ngle of the sector. Let OC
e angle AOB; the centre of
must evidently lie in this line.
e the centre of gravity, it re-
determine the distance GO.
the angles AOC, BOC, into
parts. Each half of the sector



divided into n equal sectors, and when n is very large,
y be regarded as n equal triangles, having their centres of
t a distance from $O = \frac{2}{3}r$.
he centre of gravity of each triangle with that of the
similarly placed on the other side of OC. These lines will

be at right angles to CO, and will be bisected by it. Let L, M, N, &c. be the points in which these lines cut OC, then the points L, M, N, &c. are the centres of gravity of each pair of triangles. Then, taking their distances from O, we have

$$\begin{aligned} x_1 &= \text{LO} = \frac{2r}{3}\cos\frac{a}{4n}; \\ x_2 &= \text{MO} = \frac{2r}{3}\cos\frac{3a}{4n}; \\ x_3 &= \text{NO} = \frac{2r}{3}\cos\frac{5a}{4n}; \\ &\text{\&c.} \qquad \text{\&c.} \end{aligned}$$

Hence, by Art. 372,

$$\begin{aligned} \text{GO} &= \frac{\Sigma x}{n}; \\ &= \frac{2r}{3n}\left(\cos\frac{a}{4n} + \cos\frac{3a}{4n} + \cos\frac{5a}{4n} + \text{\&}c.\ldots + \cos\frac{(2n-1)\,a}{4n}\right). \\ &= \frac{2r}{3}\cdot\frac{\sin\frac{1}{2}a}{2n\sin\frac{a}{4n}} * \end{aligned}$$

Since n is very large, $\frac{a}{4n}$ is very small, and therefore the angle itself may be substituted for its sine. Therefore,

$$\begin{aligned} \text{GO} &= \frac{2r}{3}\cdot\frac{\sin\frac{1}{2}a}{\frac{1}{2}a}; \\ &= \frac{4r}{3}\cdot\frac{\sin\frac{1}{2}a}{a}, \end{aligned}$$

the angle a being measured by the ratio of the arc to the radius.

---

* The sum of the series $\cos\phi + \cos 3\phi + \text{\&c.} + (2n-1)\,\phi$ may be easily found as follows:—

$$\begin{aligned} \sin 2\phi &= 2\cos\phi\sin\phi, \\ \sin 4\phi - \sin 2\phi &= 2\cos 3\phi\sin\phi, \\ \sin 6\phi - \sin 4\phi &= 2\cos 5\phi\sin\phi, \\ \text{\&c.} \qquad & \qquad \text{\&c.} \\ \sin 2n\phi - \sin(2n-2)\,\phi &= 2\cos(2n-1)\,\phi\sin\phi. \end{aligned}$$

Whence, by adding together,

$$\sin 2n\phi = 2\sin\phi\,\{\cos\phi + \cos 3\phi + \text{\&c.} + \cos(2n-1)\,\phi\};$$

$$\therefore \qquad \cos\phi + \cos 3\phi + \text{\&c.} + \cos(2n-1)\,\phi = \frac{\sin 2n\phi}{2\sin\phi}.$$

375. In the result just obtained, if $\alpha=\pi$, the sector is a semicircle. Therefore, if GO denote, as before, the distance of the centre of gravity from the centre,

$$\mathrm{GO}=\frac{4r}{3\pi}.$$

If $\alpha=\frac{\pi}{2}$ the sector becomes a quadrant, and in this case,

$$\mathrm{GO}=\frac{8r\sin 45^{\circ}}{3\pi}=\frac{4r\sqrt{2}}{3\pi}.$$

376. By a process precisely similar to that followed in Art. 374, t may be shown, that in the case of a circular arc subtending at he centre an angle α,

$$\mathrm{GO}=\frac{2r\sin\frac{1}{2}\alpha}{\alpha},$$

$$=\frac{\text{rad. chord}}{\text{arc}}.$$

377. The following summations will be referred to in future Articles. They may be readily found by the method of Finite Differences.

$$1+2+3+\ldots\ldots+n=\frac{n(n+1)}{2},$$

$$1^2+2^2+3^2+\ldots\ldots+n^2=\frac{n(n+1)(2n+1)}{6},$$

$$1^3+2^3+3^3+\ldots\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2,$$

$$1^4+2^4+3^4+\ldots\ldots+n^4=\frac{n(n+1)(6n^3+9n^2+n-1}{30};$$

nd hence, by making n as large as we please, the following equaons may be made as nearly true as we please:

$$\frac{1+2+3+\ldots\ldots+n}{n^2}=\frac{1}{2},$$

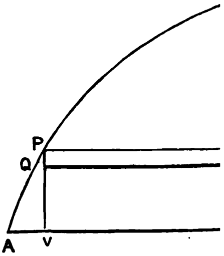
$$\frac{1^2+2^2+3^2+\ldots\ldots+n^2}{n^3}=\frac{1}{3},$$

$$\frac{1^3+2^3+3^3+\ldots\ldots+n^3}{n^4}=\frac{1}{4},$$

$$\frac{1^4+2^4+3^4+\ldots\ldots+n^4}{n^5}=\frac{1}{5}.$$

378. *To find the centre of gravity of the area of a semi-p*

Let A be the vertex and AB the axis of the parabola, and the equation of the curve.



Divide BC into n equal parts, n being an indefinitely large number, and let BC $= nk$.

Let MN be one of these divisions, then MN $= k$; and let BN $= pk$.

The area of the rectangle PM is equal to k. PN, and

$$\begin{aligned} \mathrm{PN} &= \mathrm{AB} - \mathrm{AV}, \\ &= \frac{\mathrm{BC}^2}{m} - \frac{\mathrm{PV}^2}{m}, \\ &= m^{-1}k^2(n^2 - p^2), \end{aligned}$$

$\therefore$ area of PM $= m^{-1}k^3(n^2 - p^2)$. (i.)

The distance of the centre of gravity of PM from the l is $\frac{1}{2}$ PN, or $\frac{1}{2}m^{-1}k^2(n^2 - p^2)$; therefore the product of the PM into the distance of its centre of gravity from BC is e

$$\tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2p^2 + p^4). \quad \text{(ii.)}$$

Hence, by giving to p the values 1, 2, 3, &c., in the expre we obtain

$$\begin{aligned} \mathrm{M}_1 &= m^{-1}k^3(n^2 - 1^2), \\ \mathrm{M}_2 &= m^{-1}k^3(n^2 - 2^2), \\ \mathrm{M}_3 &= m^{-1}k^3(n^2 - 3^2), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \mathrm{M}^n &= m^{-1}k^3(n^2 - n^2); \end{aligned}$$

and therefore, taking the sum

$$\begin{aligned} \Sigma.\,\mathrm{M} &= m^{-1}k^3\{n^3 - (1^2 + 2^2 + 3^2 + \ldots\ldots + n^2)\}, \\ &= m^{-1}.\,\mathrm{BC}^3\left(1 - \frac{1^2 + 2^2 + 3^2 + \ldots\ldots + n^2}{n^3}\right), \\ &= \tfrac{2}{3}m^{-1}\mathrm{BC}^3, \text{ since } n \text{ is large.} \end{aligned}$$

In like manner, from the expression ii.,

$$\begin{aligned} \mathrm{M}_1x_1 &= \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2.\,1^2 + 1^4), \\ \mathrm{M}_2x_2 &= \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2.\,2^2 + 2^4), \\ \mathrm{M}_3x_3 &= \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2.\,3^2 + 3^4), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \mathrm{M}_nx_n &= \tfrac{1}{2}m^{-2}k^5(n^4 - 2n^2.\,n^2 + n^4); \end{aligned}$$

nd therefore

$$\Sigma . \mathrm{M}x = \tfrac{1}{2}m^{-2}k^5\{n^5 - 2n^2(1^2 + 2^2 + \ldots\ldots + n^2) + (1^4 + 2^4 + \ldots\ldots n^4)\},$$

$$= \tfrac{1}{2}m^{-2}.\mathrm{BC}^5\left(1 - 2 . \frac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3} + \frac{1^4 + 2^4 + \ldots\ldots n^4}{n^5}\right),$$

$$= \tfrac{4}{15}m^{-2}\mathrm{BC}^5, \text{ since } n \text{ is large.}$$

nce, if $\bar{x}$ be the distance of the centre of gravity of the semi-rabola from BC,

$$\bar{x} = \frac{\Sigma . \mathrm{M}x}{\Sigma . \mathrm{M}},$$

$$= \tfrac{2}{5}m^{-1}\mathrm{BC}^2,$$

$$= \tfrac{2}{5}\mathrm{AB}.$$

Again, since MN is very small, the distance of the centre of vity of the rectangle PM from the line AB may be taken as al to BN or pk; therefore the product of the area of PM into distance of the centre of gravity from AB is equal to

$$m^{-1}k^4(n^2p - p^3).$$

Giving to p the values 1, 2, 3, &c., severally, we have

$$\mathrm{M}_1y_1 = m^{-1}k^4(n^2 . 1 - 1^3),$$

$$\mathrm{M}_2y_2 = m^{-1}k^4(n^2 . 2 - 2^3),$$

$$\mathrm{M}_3y_3 = m^{-1}k^4(n^2 . 3 - 3^3),$$

$$. \quad . \quad . \quad . \quad .$$

$$\mathrm{M}_ny_n = m^{-1}k^4(n^2 . n - n^3);$$

d therefore

$$\Sigma . \mathrm{M}y = m^{-1}k^4\{n^2(1 + 2 + \ldots\ldots + n) - (1^3 + 2^3 + \ldots\ldots + n^3)\},$$

$$= m^{-1}.\mathrm{BC}^4\left(\frac{1 + 2 + \ldots\ldots + n}{n^2} - \frac{1^3 - 2^3 + \ldots\ldots + n^3}{n^4}\right),$$

$$= \tfrac{1}{4}m^{-1}\mathrm{BC}^4.$$

nce, if $\bar{y}$ be the distance of the centre of gravity of the semi-rabola from the line AB,

$$\bar{y} = \frac{\Sigma . \mathrm{M}y}{\Sigma . \mathrm{M}},$$

$$= \tfrac{3}{8}\mathrm{BC}.$$

379. *To find the centre of gravity of a solid of revolution.*

Let AC be the curve which generates the solid by its revolution nd the axis AB.

Let AB be divided into n equal parts, n being an indefinitely large number, and let $AB = nh$.

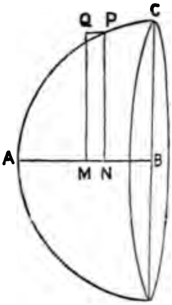
Let MN be one of these divisions, say the pth; then $MN = h$ and $AN = ph$.

The content of the cylinder generated by the revolution of the rectangle PM is equal to $\pi h . PN^2$.

Since MN is very small, the distance of the centre of gravity of this cylinder from A may be taken as equal to AN or ph; hence, if G be the centre of gravity of the solid,

$$GA = \frac{\Sigma . \pi h^2 p \, PN^2}{\Sigma . \pi h \, PN^2};$$

$$= \frac{\Sigma . ph^2 \, PN^2}{\Sigma . h \, PN^2}.$$



380. Ex. 1. To find the centre of gravity of a segment of a sphere.

Let r be the radius of the sphere, then

$$PN^2 = AN\,(2r - AN),$$
$$= 2rph - p^2h^2.$$

Hence,

$$GA = \frac{\Sigma . (2rp^2h^3 - p^3h^4)}{\Sigma . (2rph^2 - p^2h^3)};$$

$$= \frac{2rh^3(1^2 + 2^2 + \ldots + n^2) - h^4(1^3 + 2^3 + \ldots + n^3)}{2rh^2(1 + 2 + \ldots + n) - h^3(1^2 + 2^2 + \ldots + n^2)};$$

$$= \frac{2r\,AB^3 . \dfrac{1^2 + 2^2 + \ldots + n^2}{n^3} - AB^4 . \dfrac{1^3 + 2^3 + \ldots + n^3}{n^4}}{2r\,AB^2 . \dfrac{1 + 2 + \ldots + n}{n^2} - AB^3 . \dfrac{1^2 + 2^2 + \ldots + n^2}{n^3}};$$

$$= \frac{\frac{2}{3} r\,AB^3 - \frac{1}{4}\,AB^4}{r\,AB^2 - \frac{1}{3}\,AB^3},\text{*} \text{ since } n \text{ is large};$$

$$= \frac{8r . AB - 3 . AB^2}{12\,r - 4 . AB}.$$

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* From the denominator of this fraction, we see that the content of a segment of a sphere is equal to

$$\pi\,(r AB^2 - \tfrac{1}{3} AB^3).$$

Also, since $2r . AB = BC^2 + AB^2$, the content is equal to

$$\tfrac{1}{6}\,\pi AB\,(3\,BC^2 + AB^2).$$

381. COR. If the segment be a hemisphere, then $AB = r$; and therefore

$$GA = \frac{5r}{8};$$

and therefore, if O be the centre of the sphere,

$$GO = \frac{3r}{8}.$$

382. Ex. 2. To find the centre of gravity of a paraboloid.

Let m be the parameter of the generating curve, then

$$PN^2 = m \,.\, AN,$$
$$= mph.$$

Therefore,

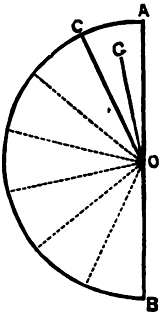
$$GA = \frac{\Sigma \,.\, mp^2h^3}{\Sigma \,.\, mph^2} = \frac{\Sigma \,.\, p^2h^3}{\Sigma \,.\, ph^2},$$

$$= \frac{h^3(1^2 + 2^2 + \ldots\ldots + n^2)}{h^2(1 + 2 + \ldots\ldots + n)},$$

$$= \frac{AB^3 . \dfrac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3}}{AB^2 . \dfrac{1 + 2 + \ldots\ldots + n}{n^2}},$$

$$= \tfrac{2}{3}AB.$$

383. *To find the centre of gravity of the portion of a sphere included between two planes intersecting in a diameter at an angle equal to any sub-multiple of two right angles.*



Let AOC be a section of the solid by a plane through the centre of the sphere, at right angles with the intersection of the planes. Then, since this plane divides the given figure symmetrically, the centre of gravity will lie in this plane.

Let the angle AOC be equal to θ, and let $n\theta = \pi$. Let G be the centre of gravity of the given solid.

Suppose the hemisphere to be divided into n equal parts, each equal and similar to the given solid. Then the sum of the moments of each part about AB is equal to the moment of the hemisphere. Hence (Art. 372),

$$\mathrm{GO}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}+\sin\frac{5\theta}{2}+\ldots\ldots+\sin\frac{(2n-1)\theta}{2}\right)=\frac{3nr}{8};$$

and therefore

$$\mathrm{GO}\times\frac{1-\cos n\theta}{2\sin\frac{1}{2}\theta}=\frac{3nr}{8}.$$

But $n\theta=\pi$, and therefore $\cos n\theta=-1$. Consequently,

$$\mathrm{GO}=\frac{3nr\sin\left(\frac{\pi}{2n}\right)}{8}.$$

384. Cor. i. Hence, generally, if θ be the angle between the planes,

$$\mathrm{GO}=\frac{3}{8}\cdot\frac{\pi r\sin\frac{1}{2}\theta}{\theta}.$$

385. Cor. ii. If the angle between the planes be indefinitely small, then, writing $\frac{\theta}{2}$ for $\sin\frac{\theta}{2}$, we have

$$\mathrm{GO}=\frac{3\pi r}{16}.$$

386. *If the mass of each of a system of particles be multiplied by the square of its distance from a given point, the sum of these products will be the least possible when the given point is the centre of gravity of the system.*

Let m_1, m_2, &c., be the masses of the particles, r_1, r_2, &c., their distances from G, the common centre of gravity, and ρ_1, ρ_2, &c., their distances from any other point O.

Let G, the centre of gravity, be taken as the origin of co-ordinates; let x_1, y_1, z_1 be the co-ordinates of m_1; x_2, y_2, z_2 those of m_2, and so on for the rest. Let a, b, c be the co-ordinates of O, then

$$\rho_1^2=(x_1-a)^2+(y_1-b^2)+(z_1-c)^2;$$
$$=x_1^2+y_1^2+z_1^2-2ax_1-2by_1-2cz_1+a^2+b^2+c^2,$$
$$\therefore\ m_1\rho_1^2=m_1r_1^2+m_1\,.\,\mathrm{GO}^2-2a\,.\,m_1x_1-2b\,.\,m_1y_1-2c\,.\,m_1z_1.$$

Similarly,

$$m_2\rho_2^2=m_2r_2^2+m_2\,.\,\mathrm{GO}^2-2a\,.\,m_2x_2-2b\,.\,m_2y_2-2c\,.\,m_2z_2,$$
$$m_3\rho_3^2=m_3r_3^2+m_3\,.\,\mathrm{GO}^2-2a\,.\,m_3x_3-2b\,.\,m_3y_3-2c\,.\,m_3z_3,$$

. . . .

Hence,

$$\Sigma . m\rho^2 = \Sigma . mr^2 + GO^2 . \Sigma m - 2a\Sigma . mx - 2b\Sigma . my - 2c\Sigma . mz.$$

But by the corollary to Art. 373,

$$\Sigma . mx = 0, \qquad \Sigma . my = 0, \qquad \Sigma . mz = 0;$$

therefore,

$$\Sigma . m\rho^2 = \Sigma . mr^2 + GO^2 \Sigma m,$$

that is, the sum of the products of each particle into the square of its distance from the centre of gravity is less than the sum of the products of each particle into the square of its distance from any other point O, by the quantity

$$GO^2 . \Sigma m;$$

or the product of the sum of the masses into the square of the distance of the point O from the centre of gravity.*

387. Cor. I. If a sphere be described, having G as its centre, and GO as its radius, then the sum of the products of each particle into the square of its distance from any point on the surface of the sphere is invariable.

388. Cor. II. If the system consist of n equal particles, $\Sigma . m\rho^2 = m\Sigma\rho^2$, $\Sigma . mr^2 = m\Sigma r^2$ and $\Sigma m = nm$. Therefore,

$$\Sigma\rho^2 = \Sigma r^2 + n . GO^2.$$

Hence, *the sum of the squares of the distances of* n *equal particles, from any point whatever, exceeds the sum of the squares of their distances from their common centre of gravity by* n *times the square of the distance of the given point from the centre of gravity.*

389. Cor. III. If three equal bodies be placed at the vertices of any triangle ABC, their centre of gravity corresponds with the centre of gravity of the triangle. Hence, if G be the centre of gravity, and O any point whatever,

$$AO^2 + BO^2 + CO^2 = AG^2 + BG^2 + CG^2 + 3 . GO^2.$$

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* If ρ_1, ρ_2, &c., denote the distances of each particle from any plane, and r_1, r_2, &c., their distances from a plane through the centre of gravity parallel to the former; then, GO being the distance between the planes, it may be easily shown that the equation

$$\Sigma . m\rho^2 = \Sigma . mr^2 + GO^2 . \Sigma m$$

still obtains.

390. COR. IV. Also, if four equal weights be placed at the vertices of a triangular pyramid, their centre of gravity corresponds with the centre of gravity of the pyramid.

Hence, if A, B, C, D be the vertices of the pyramid, G its centre of gravity, and O any point whatever,

$$AO^2 + BO^2 + CO^2 + DO^2 = AG^2 + BG^2 + CG^2 + DG^2 + 4 . GO^2.$$

391. *The sum of the products of each of a system of particles into the square of its distance from the centre of gravity of the system is equal to the sum of the products of each pair into the square of their mutual distance, divided by the sum of their particles.*

Let m_1, m_2, &c., be the particles, r_1, r_2, &c., their distances from the centre of gravity of the system, and $\rho_{1.2}$, $\rho_{2.3}$, &c., the distances of m_1 and m_2, of m_2 and m_3, &c.; then will

$$\Sigma . mr^2 = \frac{\Sigma . m_1 m_2 \rho^2_{1.2}}{\Sigma m},$$

or,

$$m_1 r_1^2 + m_2 r_2^2 + \&c. = \frac{m_1 m_2 \rho^2_{1.2} + m_1 m_3 \rho^2_{1.3} + \&c.}{m_1 + m_2 + m_3 + \&c.}.$$

Let the centre of gravity of the particles be taken as the origin of co-ordinates, and let x_1, y_1, z_1, x_2, y_2, z_2, &c., be the co-ordinates of m_1, m_2, &c.; then, by Art. 373,

$$m_1 x_1 + m_2 x_2 + \&c. = 0,$$
$$m_1 y_1 + m_2 y_2 + \&c. = 0,$$
$$m_1 z_1 + m_2 z_2 + \&c. = 0.$$

Square each of these, and add; then

$$\begin{aligned} &m_1^2(x_1^2 + y_1^2 + z_1^2) + m_2^2(x_2^2 + y_2^2 + z_2^2) + \&c. \\ &+ 2m_1 m_2(x_1 x_2 + y_1 y_2 + z_1 z_2) + 2m_1 m_3(x_1 x_3 + y_1 y_3 + z_1 z_3) \\ &+ \&c. = 0. \end{aligned}$$

But $r_1^2 = x_1^2 + y_1^2 + z_1^2$, and $r_2^2 = x_2^2 + y_2^2 + z_2^2$, and so on.

Also, $\rho^2_{1.2} = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$

$$= r_1^2 + r_2^2 - 2(x_1 x_2 + y_1 y_2 + z_1 z_2).$$

Similarly, $\rho^2_{1.3} = r_1^2 + r_3^2 - 2(x_1 x_3 + y_1 y_3 + z_1 z_3),$

&c. &c.

Substituting these values in the equation obtained above,

$$\begin{aligned} m_1^2r_1^2 + m_2^2r_2^2 + \&c. = {} & m_1m_2(\rho^2_{1\,.\,2} - r_1^2 - r_2^2), \\ & + m_1m_3(\rho^2_{1\,.\,3} - r_1^2 - r_3^2), \\ & + m_2m_3(\rho^2_{2\,.\,3} - r_2^2 - r_3^2), \\ & + \&c. \end{aligned}$$

Hence, by transposing,

$$\begin{aligned} & m_1r_1^2(m_1 + m_2 + \&c.) \\ & + m_2r_2^2(m_1 + m_2 + \&c.) \\ & + \&c. \qquad = m_1m_2\rho^2_{1\,.\,2} + m_1m_3\rho^2_{1\,.\,3} + \&c. \end{aligned}$$

or, $$m_1r_1^2 + m_2r_2^2 + \&c. = \frac{m_1m_2\rho^2_{1\,.\,2} + m_1m_3\rho^2_{1\,.\,3} + \&c}{m_1 + m_2 + m_3 + \&c.}.$$

392. Cor. i. Combining the result just obtained with that of Art. 386, we have

$$\Sigma\,.\,m\rho^2 = \frac{\Sigma\,.\,m_1m_2\rho^2_{1\,.\,2}}{\Sigma m} + GO^2\,.\,\Sigma m.$$

393. Cor. ii. If the system consist of n equal particles, then the result of Art. 391 becomes

$$\Sigma\,.\,\rho^2_{1\,.\,2} = n\,.\,\Sigma\,.\,r^2.$$

Hence, *the sum of the squares of the mutual distances of each pair out of* n *equal particles is equal to* n *times the sum of the squares of the distances of the particles from their common centre of gravity.*

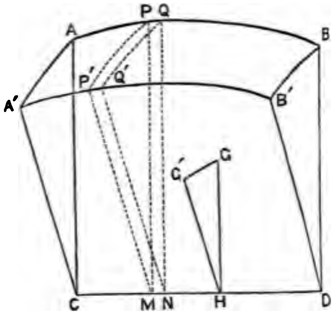
394. Cor. iii. Consequently, (see Art. 389,) if a, b, c be the sides of any triangle, and k, l, m the distances from the centre of gravity, $$a^2 + b^2 + c^2 = 3\,(k^2 + l^2 + m^2).$$

395. Cor. iv. Also, (see Art. 390,) if a, b, c, d, e, f be the edges of any triangular pyramid, and k, l, m, n the distances of the vertices from the centre of gravity, then

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 4\,(k^2 + l^2 + m^2 + n^2).$$

GULDIN'S PROPERTIES.

396. *If any plane figure revolve about an axis lying in its own plane, the content of the generated solid is equal to the product of the generating area into the length of the arc described by its centre of gravity.*



Let ABDC be the generating area, and let it revolve about the axis CD, through any angle θ, into the position A′B′DC.

Let CD be divided into n equal parts, n being an indefinitely large number, and let CD $= nh$.

Let MN be any one of these divisions, then MN $= h$. Through M and N draw PM and QN, at right angles with CD; then PN may be regarded as a rectangle, and the content of the solid generated by the revolution of PN is equal to

$$h \times \text{area of the sector PMP}'.$$

Let PM $= y$, then if θ be measured by the ratio of the subtended arc to the radius, the area of the sector PMP′ is equal to $\frac{1}{2}y^2\theta$; hence, the content of the solid generated by the revolution of PN is equal to

$$\tfrac{1}{2}y^2\theta h,$$

and therefore

$$\text{content of solid} = \tfrac{1}{2}\Sigma\,.\,y^2\theta h\,;$$
$$= \tfrac{1}{2}\theta\Sigma\,.\,y^2h.$$

Again, the area of the rectangle PN is equal to yh, and the distance of its centre of gravity from CD is equal to $\frac{1}{2}y$, therefore the moment of PN about CD is equal to

$$\tfrac{1}{2}y^2h.$$

Hence, if $\bar{y}$ be the distance of the centre of gravity of the area ABDC from CD, then

$$\bar{y} = \frac{\frac{1}{2}\Sigma\,.\,y^2h}{\Sigma\,.\,yh}.$$

Therefore,

$$\text{content of solid} = \bar{y}\theta \times \Sigma\,.\,yh.$$

But if G be the centre of gravity of the area ABDC, the arc $GG' = \bar{y}\theta$ and $\Sigma . yh$ is the area of ABDC; therefore,

$$\text{content of solid} = GG' \times \text{area of ABDC}.$$

397. *If any plane curve revolve about an axis lying in its own plane, the area of the generated surface is equal to the product of the length of the curve into the length of the arc described by its centre af gravity.*

Let AB be the curve, and let it revolve about the axis CD, through any angle θ, into the position A′B′.

Let AB be divided into n equal parts, n being an indefinitely large number, and let $AB = ns$.

Let PQ be one of these parts, then $PQ = s$, and since PQ is very small, it may be regarded as a straight line; and if $PM = y$, the area generated by the revolution of PQ is equal to

$$y\theta . s,$$

and hence,

$$\text{area of } ABB'A' = \Sigma . y\theta s,$$
$$= \theta\Sigma . ys.$$

Also, since PQ is small, the distance of its centre of gravity from CD may be taken as equal to y, and therefore the moment of PQ about CD is equal to ys;

and hence, if $\bar{y}$ be the distance of the centre of gravity of AB from the line CD,

$$\bar{y} = \frac{\Sigma . ys}{AB},$$

and therefore

$$\text{area of } ABB'A' = \bar{y}\theta . AB.$$

If G be the centre of gravity of AB, then $GG' = \bar{y}\theta$, and therefore

$$\text{area of } ABB'A' = GG' \times AB.$$

EXAMPLES.

1. Find the centre of gravity of a segment of a circle; show that the distance of the centre of gravity from the centre of the circle is equal to the cube of the chord divided by 12 times the area of the segment.

2. Find the centre of gravity of the crescent formed by two arcs of equal circles, when the centre of one circle is upon the circumference of the other.

Let r be the radius, then the distance of the centre of gravity from the middle of the common chord of the two arcs is equal to

$$\frac{3\pi r}{2\pi + 3\sqrt{3}}.$$

3. Find the centre of gravity of a spherical sector.

Let the sector be divided into a spherical segment and a cone, let r be the radius of the sphere and x the thickness of the segment, then the distance of the centre of gravity from the middle of the spherical surface of the sector is equal to

$$\frac{2r + 3x}{8}.$$

4. Find the centre of gravity of one-eighth of a sphere bounded by three planes at right angles to one another.

Show that the distance of the centre of gravity from the centre of the sphere is $\frac{3r\sqrt{3}}{8}$.

5. If a regular polygon of n sides be inscribed in a great circle of a sphere, the sum of the squares of the distances of the angular points of the polygon from any point on the sphere is equal to $2n$ times the square of the radius of the sphere.

6. The sum of the squares of the mutual distances of each pair of angular points in a regular hexagon is equal to the square of the periphery of the hexagon.

7. In a regular polygon of n sides, the sum of the squares of the distances of any one angular point from the rest is equal to $2n$ times the square of the radius of the circumscribing circle.

8. Find the centre of gravity of the portion of a sphere contained between two parallel planes, one of which passes through

the centre of the sphere; the distance between the planes being equal to half the radius.

Let r be the radius of the sphere, then the distance of the centre of gravity from the centre of the sphere is equal to

$$\frac{21r}{88}.$$

9. Assuming that the area of an ellipse is the product of its semi-axes multiplied by π, find the centre of gravity of an elliptic quadrant.

Let a and b be the semi-axes, then the distance of the centre of gravity from the major-axis is equal to

$$\frac{4b}{3\pi},$$

and its distance from the minor-axis is equal to

$$\frac{4a}{3\pi},$$

10. Find the centre of gravity of the segment of a prolate spheroid cut off by a plane at right angles with the axis of revolution.

Let x be the length of the axis of the segment, and a the semi-major axis, then the distance of the centre of gravity from the vertex is equal to

$$\frac{8ax - 3x^2}{12a - 4x}.$$

11. Find the centre of gravity of the area contained between a parabolic arc AQP, and a chord AP drawn through the vertex.

Let C be the middle part of the chord AP, and let a parallel to the axis through C meet the curve in Q; show that the centre of gravity is in the line CQ, at a distance from C equal to

$$\frac{2CQ}{5}.$$

12. Apply Guldin's Properties to the determination of the content and surface of a ring.

Let a be the radius of the generating circle, and b the distance of its centre from the axis of revolution, then the content of the ring is equal to $2\pi^2a^2b$, and the surface of the ring is equal to $4\pi^2ab$.

13. Find, by Guldin's Properties, the content of an oblate spheroid.

Let a and b be the semi-axes of the generating ellipse, the content required is equal to $\frac{4}{3}\pi a^2 b$.

14. The surface of a sphere whose radius is r is equal to $4\pi r^2$; determine the centre of gravity of a semi-circular arc.

15. Find the content of the solid generated by the revolution of a triangle about one of its sides.

Let a, b, and c be the sides of the triangle, and let c be the axis of revolution, then the required content is equal to

$$\frac{4\pi s\,.\,(s-a)\,(s-b)\,(s-c)}{3c}.$$

16. Show that the content of the solid generated by the revolution of a segment of a circle about the diameter which is parallel to its chord is equal to $\frac{1}{6}\pi \times (\text{chord})^3$.

17. Find the content of the spindle generated by the revolution about its chord of a circular segment whose height is equal to half the radius.

Let r be the radius of the segment, then the required content is equal to

$$\pi r^3\left(\frac{3\sqrt{3}}{4}-\frac{\pi}{3}\right).$$

18. Find the area of a spherical zone.

Let r be the radius of the sphere, and c the axis of the zone, then the required area is equal to $2\pi cr$.

19. Assuming the preceding, show that the centre of gravity of any spherical zone, or of the surface of any spherical segment, is the middle point of the axis.

20. Show, from Guldin's Properties, that the content of a paraboloid of revolution is equal to the area of its plane surface multiplied by half its axis.

(For the area of a semi-parabola, refer to Art. 378.)

HYDROSTATICS.

CHAPTER V.

ON THE CENTRE [illegible]SSURE.

398. The pressure of a fluid up[illegible] surface varies, as already seen, with the depth of each point below the surface of the fluid. The point in the surface through which the resultant of these several pressures passes is called the *centre of pressure.* When the surface in contact with the fluid is a plane surface, the pressures at the different points are all perpendicular to the surface, and hence are parallel forces. Hence, if the pressures on the several parts of a surface be P_1, P_2, &c., and k_1, k_2, &c., be the distances of the centres of pressure of these parts from the surface of the fluid, and k the distance of the centre of pressure of the whole plane from the surface of the fluid (Art. 49),

$$k = \frac{P_1 k_1 + P_2 k_2 + \&c.}{P_1 + P_2 + \&c.}.$$

But if m_1, m_2, &c., be the areas of the several parts of the plane, and h_1, h_2, &c., the distances of their *centres of gravity* below the surface of the fluid, and w the weight of a unit of the fluid, $P_1 = wm_1h_1$, $P_2 = wm_2h_2$, &c.;

$$\therefore \qquad k = \frac{m_1 h_1 k_1 + m_2 h_2 k_2 + \&c.}{m_1 h_1 + m_2 h_2 + \&c.}.$$

If the parts into which the plane is divided be indefinitely small rectangles, having one pair of sides parallel to the surface of the fluid, the centres of pressure and the centres of gravity may be regarded as coincident. Whence,

$$k = \frac{m_1 h_1^2 + m_2 h_2^2 + \&c.}{m_1 h_1 + m_2 h_2 + \&c.},$$

or, as it may be conveniently written,

$$k = \frac{\Sigma . mh^2}{\Sigma . mh}.$$

Cor. If the areas of the several parts are equal, the last expression becomes

$$k = \frac{h_1^2 + h_2^2 + \&c.}{h_1 + h_2 + \&c.} = \frac{\Sigma . h^2}{\Sigma . h}.$$

399. *To find the centre of pressure of any parallelogram, one of whose sides is coincident with the surface of the fluid.*

Let h be the depth of the lower side of the parallelogram. Let the parallelogram be divided by lines parallel to the surface into n equal parallelograms, n being an indefinitely large number. Since these parallelograms are indefinitely small, the depth of their centres of gravity may be taken as equal to that of their lower sides. Hence,

$$k = \frac{\left(\frac{h}{n}\right)^2 + \left(\frac{2h}{n}\right)^2 + \left(\frac{3h}{n}\right)^2 + \ldots\ldots + \left(\frac{nh}{n}\right)^2}{\frac{h}{n} + \frac{2h}{n} + \frac{3h}{n} + \ldots\ldots + \frac{nh}{n}},$$

$$= h \frac{\frac{1 + 2^2 + \ldots\ldots + n^2}{n^3}}{\frac{1 + 2 + \ldots\ldots + n}{n^2}} = \frac{2h}{3}. \text{ (Art. 377.)}$$

or the centre of pressure is at two-thirds of the depth.

Hence, the staves of a cylindrical barrel, containing any liquid, may be kept in their place by a single hoop placed at two-thirds of the depth of the fluid.

400. *To determine the centre of pressure of a parallelogram immersed to any depth, but having one of its sides parallel to the fluid*

t ABCD be any parallelogram, having its lower side AB lel to the surface of the fluid. Let h be the depth of AB r the surface, and h' the depth of the opposite side DC. uce the sides AD, BC to meet the surface of the fluid in .d L. Let $AB = a = CD$, and let θ be the inclination of the › to the vertical line.

$$\text{Area of ABLK} = \frac{ah}{\cos\theta};$$

$$\therefore \qquad \text{pressure on ABLK} = \frac{ah}{\cos\theta} \cdot \frac{h}{2}.$$

.arly, $$\text{pressure on DCLK} = \frac{ah'}{\cos\theta} \cdot \frac{h'}{2}.$$

e, if k be the depth of the centre of pressure on ABCD,

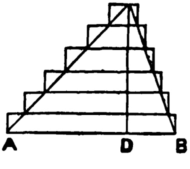
$$k = \frac{\dfrac{ah}{\cos\theta} \cdot \dfrac{h}{2} \cdot \dfrac{2h}{3} - \dfrac{ah'}{\cos\theta} \cdot \dfrac{h'}{2} \cdot \dfrac{2h'}{3}}{\dfrac{ah}{\cos\theta} \cdot \dfrac{h}{2} - \dfrac{ah'}{\cos\theta} \cdot \dfrac{h'}{2}},$$

$$= \frac{2}{3} \cdot \frac{h^3 - h'^3}{h^2 - h'^2},$$

$$= \frac{2}{3} \cdot \frac{h^2 + hh' + h'^2}{h + h'}.$$

1. *To find the centre of pressure of a triangular plane, whose ; is on the surface of the fluid, and whose base is parallel to ırface of the fluid.*

t ABC be the triangle, and AB its base. Draw CD perpen-ır to the base, and divide it into n equal Through these points of division draw parallel to AB, and upon these lines as describe rectangles, as in the figure. increases, the sum of the pressures on ectangles approaches to the pressure on iangle.



Let AB $= a$, and let h be the depth of AB below the surface of the fluid. Then the base of the first rectangle is equal to $\frac{a}{n}$, the base of the second rectangle to $\frac{2a}{n}$, and so on;

$$\therefore \quad \text{area of first rectangle} = \frac{a}{n} \cdot \frac{\text{CD}}{n},$$

$$\text{second} \quad ,, \quad = \frac{2a}{n} \cdot \frac{\text{CD}}{n},$$

$$\text{third} \quad ,, \quad = \frac{3a}{n} \cdot \frac{\text{CD}}{n},$$

$$\&\text{c.} \qquad \&\text{c.}$$

The depth of the centre of gravity of the first rectangle may be taken as equal to $\frac{h}{n}$, of the second $\frac{2h}{n}$, of the third $\frac{3h}{n}$, and so on.

Let k be the depth of the centre of pressure of the triangle, then

$$k = \frac{\dfrac{a\text{CD}}{n^2} \cdot \left(\dfrac{h}{n}\right)^2 + \dfrac{2a\text{CD}}{n^2} \cdot \left(\dfrac{2h}{n}\right)^2 + \&\text{c.} + \dfrac{na\text{CD}}{n^2} \left(\dfrac{nh}{n}\right)^2}{\dfrac{a\text{CD}}{n^2} \cdot \dfrac{h}{n} + \dfrac{2a\text{CD}}{n^2} \cdot \dfrac{2h}{n} + \&\text{c.}. + \dfrac{n \,.\, a\text{CD}}{n^2} \cdot \dfrac{nh}{n}},$$

$$= h \frac{\dfrac{1^3 + 2^3 + \ldots\ldots + n^3}{n^4}}{\dfrac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3}} = \frac{3h}{4} \quad \text{(Art. 377.)}$$

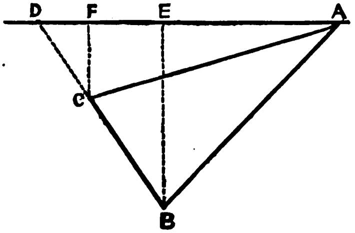
402. *To find the centre of pressure of a triangular plane whose base is on the surface of the fluid.*

Let h be the depth of the vertex, k the depth of the centre of pressure. The given triangle is the half of a parallelogram, one of whose sides is coincident with the surface of the fluid, and whose opposite side is at depth h. The centre of pressure of this parallelogram is known from Art. 399. The other half of the parallelogram is a triangle whose vertex is on the surface, and whose base is parallel to the surface; its centre of pressure is therefore known by the preceding Article. Then, if A be the area of the given triangle,

$$k = \frac{\frac{2Ah}{2} \cdot \frac{2h}{3} - \frac{2Ah}{3} \cdot \frac{3h}{4}}{\frac{2Ah}{2} - \frac{2Ah}{3}} = h \cdot \frac{\frac{1}{3} - \frac{1}{4}}{\frac{1}{2} - \frac{1}{3}} = \tfrac{1}{2}h.$$

403. *To find the centre of pressure of any triangular plane whose vertex is on the surface of the fluid.*

Let ABC be the given plane, and let h_1, h_2 be the depths of B and C respectively.



Let BC produced meet the surface of the fluid in D. Draw BE, CF, perpendicular to AD. Then, if α be the inclination of the plane of ABC to the surface of the fluid,

$$\text{BE} \sin \alpha = h_1,$$

and $$\text{CF} \sin \alpha = h_2.$$

The given plane is equal to the difference of the two triangles ABD and ACD.

The area of ABD is equal to $\frac{1}{2}$ AD × BE, or $\frac{\text{AD} \cdot h_1}{2 \sin \alpha}$; the depth of its centre of gravity is $\frac{1}{3}h_1$; and the depth of its centre of pressure is (Art. 402) $\frac{1}{2}h_1$.

The area of ACD is equal to $\frac{1}{2}$ AD × CF, or $\frac{\text{AD} \cdot h_2}{2 \sin \alpha}$; the depth of its centre of gravity is $\frac{1}{3}h_2$; and the depth of its centre of pressure is $\frac{1}{2}h_2$.

Therefore, if k be the depth of the centre of pressure of ABC,

$$k = \frac{\frac{\text{AD} \cdot h_1}{2 \sin \alpha} \cdot \frac{h_1}{3} \cdot \frac{h_1}{2} - \frac{\text{AD} \cdot h_2}{2 \sin \alpha} \cdot \frac{h_2}{3} \cdot \frac{h_2}{2}}{\frac{\text{AD} \cdot h_1}{2 \sin \alpha} \cdot \frac{h_1}{3} - \frac{\text{AD} \cdot h_2}{2 \sin \alpha} \cdot \frac{h_2}{3}},$$

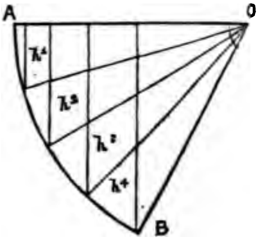
$$= \tfrac{1}{2} \cdot \frac{h_1^3 - h_2^3}{h_1^2 - h_2^2},$$

$$= \tfrac{1}{2} \cdot \frac{h_1^2 + h_1 h_2 + h_2^2}{h_1 + h_2}.$$

COR. If h_2 is nearly equal to h_1, the depth of the centre of pressures is nearly $\dfrac{3h_1}{4}$.

404. *To find the depth of the centre of pressure of a circular sector immersed vertically with one of its radii upon the surface of the fluid.*

Let AOB be the sector, having the angle AOB equal to α. Divide the sector into n equal sectors, n being indefinitely large. These sectors may be regarded as small triangles.



Let h_1 be the depth of the lowest vertex of the first triangle, h_2, that of the second, and so on. Then the depths of the centre of gravity of the different triangles are respectively

$$\frac{2h_1}{3},\quad \frac{2h_2}{3}, \ldots\ldots \frac{2h_n}{3},$$

and the depths of the centres of pressure (Art. 403, Cor.)

$$\frac{3h_1}{4},\quad \frac{3h_2}{4}, \ldots\ldots \frac{3h_n}{4}.$$

Therefore, if k be the depth of the centre of pressure of the sector,

$$k = \frac{\frac{1}{2}\left(h_1^2 + h_2^2 + \ldots\ldots + h_n^2\right)}{\frac{2}{3}\left(h_1 + h_2 + \ldots\ldots + h_n\right)}.$$

But if a be the radius of the sector, $h_1 = a \sin\dfrac{\alpha}{n}$, $h_2 = a \sin\dfrac{2\alpha}{n}$, and so on.

Therefore, substituting these values,

$$k = \frac{3a}{4} \cdot \frac{\sin^2\dfrac{\alpha}{n} + \sin^2\dfrac{2\alpha}{n} + \ldots\ldots + \sin^2\dfrac{n\alpha}{n}}{\sin\dfrac{\alpha}{n} + \sin\dfrac{2\alpha}{n} + \ldots\ldots + \sin\dfrac{n\alpha}{n}},$$

$$= \frac{3a}{8} \cdot \frac{n - \left(\cos\dfrac{2\alpha}{n} + \cos\dfrac{4\alpha}{n} + \ldots + \cos\dfrac{2n\alpha}{n}\right)}{\sin\dfrac{\alpha}{n} + \sin\dfrac{2\alpha}{n} + \ldots\ldots + \sin\dfrac{n\alpha}{n}},$$

$$= \frac{3a}{8} \cdot \frac{n - \dfrac{\cos\left(\dfrac{n+1}{n}\cdot\alpha\right)\sin\alpha}{\sin\dfrac{\alpha}{n}}}{\dfrac{\sin\left(\dfrac{n+1}{n}\cdot\dfrac{\alpha}{2}\right)\sin\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2n}}},$$

$$= \frac{3a}{8} \cdot \frac{\alpha - \cos\alpha\sin\alpha}{1 - \cos\alpha}, \text{ since } n \text{ is large.}$$

405. **Cor.** I. If the sector be a quadrant, $\alpha = \frac{\pi}{2}$, and therefore

$$k = \frac{3\pi a}{16}.$$

406. **Cor.** II. If the sector be a semi-circle, $\alpha = \pi$, and therefore

$$k = \frac{3\pi a}{16}.$$

407. **Cor.** III. Hence, also, if a sector be immersed with its plane vertical, its centre upon the surface, and its axis inclined to the surface at an angle θ; then α being the angle of the sector,

$$k = \frac{3a}{16} \cdot \frac{2\alpha - \sin(2\theta + \alpha) + \sin(2\theta - \alpha)}{\cos(\theta - \frac{\alpha}{2}) - \cos(\theta + \frac{\alpha}{2})}.$$

408. **Cor.** IV. If a semi-circle, having its diameter on the surface of a fluid, be divided into small rectangles by lines parallel to the surface, and if m denote the area of one of these, and h the depth of its centre of gravity, then (Art. 398),

$$k = \frac{\Sigma \,.\, mh^2}{\Sigma \,.\, mh}.$$

But, $k = \frac{3\pi a}{16}$, and $\Sigma \,.\, mh$, or the sum of the moments of the several rectangles about the diameter is equal to the moment of the whole semi-circle; that is, to $\frac{\pi a^2}{2} \times \frac{4a}{3\pi}$ or $\frac{2}{3}a^3$. Substituting these values, we have

$$\Sigma \,.\, mh^2 = \frac{\pi a^4}{8}.$$

In like manner, if h_2, k_2 be the distances of the centre of gravity and the centre of pressure from the surface line in the second position,

$$h_2(k_2 - h_2) = \frac{\Sigma . mr^2}{\Sigma m}. \qquad \text{(ii.)}$$

But, by hypothesis, the line through the centre of gravity, parallel to the surface of the fluid, is the same in both cases, and consequently $\Sigma . mr^2$ is the same in equations i. and ii.; therefore,

$$h_2\,(k_2 - h_2) = h_1\,(k_1 - h_1).$$

COR. If the figure is symmetrical about an axis, and the axis is at right angles to the surface line, then since both the centre of gravity and the centre of pressure are in the axis, the distance $k - h$ is in each case the distance of pressure from the centre of gravity, and hence the preceding theorem shows, that *if any symmetrical figure be immersed with its axis at right angles to the surface line, the product of the distance of the centre of gravity from the surface line into the distance of the centre of pressure from the centre of gravity is invariable.*

EXAMPLES.

1. Find the centre of pressure of a triangular plane immersed to any depth, having its base parallel to the surface of the fluid, and lower than the vertex.

Let c be the depth of the vertex, and $c + h$ that of the base; then the depth of the centre of pressure is equal to

$$\frac{6c^2 + 8ch + 3h^2}{6c + 4h}.$$

2. Find the centre of pressure of a square plane, having one of its corners upon the surface of the fluid, and one of its diagonals vertical.

Let a be the side of the square, then the depth of the centre of pressure is equal to $\dfrac{7a\sqrt{2}}{12}$.

3. Find the centre of pressure of a hexagonal plane immersed vertically, with one of its sides upon the surface of the fluid.

Let a be a side of the hexagon, then the depth of the centre of pressure is equal to
$$\frac{23a\sqrt{3}}{36}.$$

4. Find the centre of pressure of a trapezium immersed with its base upon the surface of the fluid, and its parallel sides perpendicular to the surface.

Let h_1 and h_2 be the lengths of the parallel sides, then the depth of the centre of pressure is equal to
$$\frac{1}{2}\frac{(h_1+h_2)(h_1^2+h_2^2)}{h_1^2+h_1h_2+h_2^2}.$$

5. A conical vessel, filled with a liquid, is placed with its vertex upwards, and its axis inclined to the vertical line at an angle equal to half the angle of the cone; determine the distance of the centre of pressure of the base from the centre of the base.

Let 2α be the angle of the cone, and r the radius of the base; then the distance required is equal to
$$\frac{r\tan^2\alpha}{4}.$$

6. An octagonal plate is immersed in a fluid, so that one of its sides is upon the surface; find the distance of the centre of pressure from the centre of the plate.

Let a be the side of the plate, then the distance required is equal to
$$\frac{a(1+2\sqrt{2})}{12}.$$

CHAPTER VI.

ON THE EQUILIBRIUM OF FLOATING BODIES.

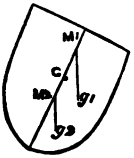
411. It has already been seen that, when a floating body is in equilibrium, the weight of the body acting vertically downwards at its centre of gravity is equal and opposite to the weight of the fluid displaced, acting at the point which was the centre of gravity of this portion of the fluid before displacement. If then a vertical line passing through this last-mentioned point be called the *line of thrust*, one of the conditions of the equilibrium of a floating body will be, that the line of thrust pass through its centre of gravity.

412. A body at rest is said to be in stable equilibrium, if after receiving a slight displacement it tends to return to its former position, but *unstable* if it tends to recede from it. If neither of these happen—that is, if equilibrium be not destroyed by the displacement—the body is said to be in neutral equilibrium.

413. If a body in equilibrium be displaced through an indefinitely small angle, the point in which the lines of thrust before and after displacement intersect one another is termed the *metacentre*.

414. *The equilibrium of a floating body is stable, unstable, or neutral, according as the metacentre is above, below, or coincident with the centre of gravity.*

Let G be the centre of gravity of a floating body, and M_1GM_2 the line of thrust when at rest. Let g_1M_1 be the line of thrust after displacement, then M_1 is the metacentre, and falls above G. The body is then acted upon by two equal forces, the weight of the body acting vertically downwards at G, and the resultant pressure of the fluid acting upwards at g_1.



These forces will rise to a motion of rotation, such as tends to restore the line M_1GM_2 to the vertical position, and consequently the equilibrium is stable.

Again, let g_2M_2 be the line of thrust after displacement, so that the metacentre is below G, then the two forces acting upon the body tend to produce rotation in a direction contrary to that in the former case, or to cause M_1GM_2 to recede from the vertical position; the equilibrium is therefore unstable.

If the line of thrust after displacement pass through G, the body will still remain at rest, and therefore the equilibrium is neutral.

415. *To find the depth of the immersed part of any floating body when its form is similar to that of the whole body.*

Let V be the volume of the given body, V′ that of the part immersed, and s the ratio of the specific gravity of the body to that of the fluid. Let w be the weight of a unit of the fluid, then the weight of the body is equal to

$$Vsw,$$

and the weight of the fluid displaced is equal to

$$V'w;$$

and therefore, by Art. 289,

$$V'w = Vsw,$$

and hence,

$$\frac{V'}{V} = s.$$

Let h be the vertical height of the floating body, and h' that of the part immersed. Then, if the body be a plate of uniform thickness,

$$\frac{h'^2}{h^2} = \frac{V'}{V} = s.$$

and therefore, in this case, $h' = h\sqrt{s}$.

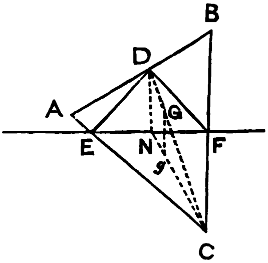
Secondly: if the body be not of uniform thickness, then, since by hypothesis the floating body and the part immersed are similar solids,

$$\frac{h'^3}{h^3} = \frac{V'}{V} = s.$$

and therefore, in this case, $h' = h\sqrt[3]{s}$.

416. *To find the positions of equilibrium of a triangular plate floating vertically in any fluid.*

Let ABC be the triangular plate, G its centre of gravity, and s the ratio of the specific gravity of the floating body to that of the fluid.



First: let one vertex only be immersed, as in the figure, and let g be the centre of gravity of the displaced fluid. Then, by the conditions of equilibrium, gG must be vertical, and the weight of the fluid displaced must equal the weight of the floating body.

Let $AC = a$, $BC = b$, $EC = x$, and $FC = y$, and let C represent the angle ACB;

$$\text{area of ABC} = \tfrac{1}{2}ab \sin C.$$
$$\text{area of EFC} = \tfrac{1}{2}xy \sin C.$$

Hence, if w be the weight of a unit of the fluid, and s the ratio of the specific gravity of the body to that of the fluid,

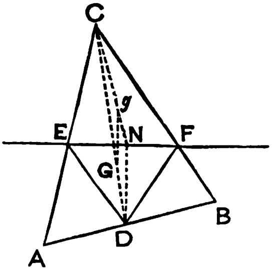
$$\text{weight of ABC} = \tfrac{1}{2}\, swab \sin C,$$
$$\text{weight of displaced fluid} = \tfrac{1}{2}\, wxy \sin C.$$

Therefore,
$$wxy = swab,$$
$$xy = sab. \qquad \text{(i.)}$$

Since g is the centre of gravity of EFC, N is the bisection of EF, and $Ng = \frac{1}{3}$ NC. In like manner, D is the bisection of AB, and $DG = \frac{1}{3}$ DC. Hence, in the triangle DCN, $DG : DC :: Ng : NC$, and therefore DN and Gg are parallel. Consequently, DN is vertical, or at right angles to EF, and therefore the lines DE and DF are equal,

$$\text{But} \qquad DE^2 = DC^2 + x^2 - 2xDC \cos ACD,$$
$$\text{and} \qquad DF^2 = DC^2 + y^2 - 2yDC \cos BCD;$$
$$\therefore \quad x^2 - 2xDC \cos ACD = y^2 - 2yDC \cos BCD. \qquad \text{(ii.)}$$

The equations i. and ii. are sufficient for determining the values of x and y, or the positions of equilibrium, when C only is immersed. And, in like manner, the positions of equilibrium may be found when either of the vertices is immersed singly.



Secondly; let one vertex only, C, be without the fluid. Then, as before, Gg must be vertical; for since G is the centre of gravity of the whole figure ABC, and g of the part EFC, the centre of gravity of the remainder ABFE must lie in the line joining Gg. Consequently, if, as before, $\text{CE} = x$, and $\text{CF} = y$, equation ii. applies to this case also. To obtain the other equation,

$$\text{ABFE} = \text{ABC} - \text{EFC},$$
$$= \tfrac{1}{2}(ab - xy) \sin \text{C};$$
$$\therefore \quad \text{weight of displaced fluid} = \tfrac{1}{2}w\,(ab - xy) \sin \text{C};$$
$$\therefore \quad w(ab - xy) \sin \text{C} = swab \sin \text{C},$$
$$ab - xy = sab,$$
$$xy = (1 - s)ab.$$

417. The equations deduced in the preceding Article admit of easy solution, if the vertex to which they apply be the vertex of an isosceles triangle. Let this angle be 2α, and let each of the equal sides be a; then $\text{DC} = a \cos \alpha$, and the equations i. and ii. become

$$xy = sa^2,$$
$$\text{and} \quad x^2 - 2ax \cos^2 \alpha = y^2 - 2ay \cos^2 \alpha;$$
$$\therefore \quad x^2 - y^2 - 2a \cos^2 \alpha\,(x - y) = 0,$$
$$\therefore \quad (x - y)\,(x + y - 2a \cos_2 \alpha) = 0.$$

This equation is satisfied when either of its factors vanishes. Let the first factor vanish, then

$$x - y = 0;$$
$$\therefore \quad x = y = a\sqrt{s},$$

which gives the position of the triangle when floating with DC vertical. Let the second factor vanish, then

$$x + y - 2a \cos^2 \alpha = 0.$$

Substituting the value of y, from the equation $xy = sa^2$,

$$x^2 - 2ax \cos^2 \alpha + sa^2 = 0;$$

$$\therefore \qquad x = a \cos^2 \alpha \pm a\sqrt{(\cos^4 \alpha - s)}.$$

If s be greater than $\cos^4 \alpha$, this expression involves an impossible quantity, and the only position of equilibrium will be that already determined. If $s = \cos^4 \alpha$, the expression under the radical vanishes, and we have

$$x = a \cos^2 \alpha = a\sqrt{s};$$

$$\therefore \qquad y = \frac{s \, . \, a^2}{a \, . \cos^2 \alpha} = a\sqrt{s},$$

which gives the same position as before. If s be less than $\cos^4 \alpha$, the equations will give two real values for x and y, and there may consequently be two positions of equilibrium besides that already determined.

If the vertex be without the fluid, the equations for the determination of the position of equilibrium become

$$xy = (1 - s)\, a^2,$$

and $$(x - y)\,(x + y - 2a \cos^2 \alpha) = 0.$$

Whence $$x = a\sqrt{(1 - s)},$$

and $$x = a \cos^2 \alpha \pm a\sqrt{\{\cos^4 \alpha - (1 - s)\}},$$

which may give three positions of equilibrium, if $1 - s$ be less than $\cos^4 \alpha$; that is, if s be greater than $1 - \cos^4 \alpha$.

418. *To find the greatest number of positions in which an equilateral triangular plate will float in a given fluid.*

Since the plate is equilateral, $\alpha = 30°$, and therefore $\cos^2 \alpha = \frac{3}{4}$. Let $x_1, x_2, x_3, y_1, y_2, y_3$ be the different possible values of x and y when any one vertex is immersed; then

$$x_1 = a\sqrt{(s)} = y_1,$$
$$x_2 = a\left\{\tfrac{3}{4} + \sqrt{(\tfrac{9}{16} - s)}\right\} = y_3,$$
$$x_3 = a\left\{\tfrac{3}{4} - \sqrt{(\tfrac{9}{16} - s)}\right\} = y_2.$$

In order that x_2, x_3, y_2, y_3 may be possible, s must be less than $\frac{9}{16}$. The nature of the problem also requires that neither x nor y shall be greater than a. No value below $\frac{9}{16}$ given to s will make either the first or third expression greater than a. That the second expression be not greater than a, s must be so taken that

$\frac{3}{4}+\sqrt{(\frac{9}{16}-s)}$ is not greater than 1,
or, $\frac{9}{16}-s$ „ „ $\frac{1}{16}$,
or, s is not less than $\frac{1}{2}$.

If s be less than $\frac{1}{2}$, x_2 and y_3 are both greater than a, and therefore the positions denoted by x_2, y_2, and by x_3, y_3, are inadmissible.

Again, let x_4, y_4, x_5, y_5, x_6, y_6, be the possible values of x and y, when any vertex is singly above the fluid. Then

$$x_4 = a\sqrt{(1-s)} = y_4,$$
$$x_5 = a\left\{\frac{3}{4}+\sqrt{(s-\frac{7}{16})}\right\} = y_6,$$
$$x_6 = a\left\{\frac{3}{4}-\sqrt{(s-\frac{7}{16})}\right\} = y_5.$$

In order that the second and third of these expressions may be possible, s must be greater than $\frac{7}{16}$, and that the second may not be greater than a, s must not be greater than $\frac{1}{2}$. If s be greater than $\frac{1}{2}$, x_5 and y_6 are both greater than a, and therefore the positions denoted by x_5, y_5, and by x_6, y_6, are inadmissible.

Hence we obtain the following results:—

First: If s be greater than unity, the body will not float at all.

Secondly: If s lie between 1 and $\frac{9}{16}$ the body will float in six positions; namely, when either vertex is singly above or below the surface, and the opposite side horizontal.

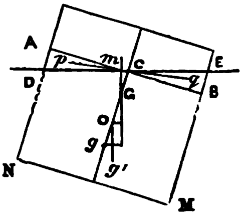
Thirdly: If s lie between $\frac{9}{16}$ and $\frac{1}{2}$, each vertex, when singly immersed, will rest in three positions, and when singly above the surface in one; namely, with the opposite side horizontal; there will therefore be twelve positions.

Fourthly: If s lie between $\frac{1}{2}$ and $\frac{7}{16}$, each vertex, when singly above the surface, will rest in three positions, and when singly below in one; namely, with the base horizontal; there are therefore twelve positions of equilibrium.

Fifthly: If $s=\frac{1}{2}$, there are also twelve positions only. For though it might seem as if this would give three values for x, both when each vertex is singly below the surface and when above, it will be seen that when s has this value, either x or y, except when one side is horizontal, is equal to a, and consequently one vertex is on the surface. Six of the positions, therefore, having one vertex singly below will be identical with six having one vertex singly above. There will not then be eighteen positions of equilibrium, but twelve only.

Sixthly: If s be less than $\frac{7}{18}$, the body will float in six positions; namely, with either vertex singly above or below the surface, and the base horizontal.

419. *A square plate of uniform thickness and density floats vertically in a fluid with one side horizontal; to determine the position of the metacentre, and the nature of the equilibrium.*



Let $2a$ be the length of a side of the plate, and s the specific gravity of the plate relatively to that of the fluid. Let ABMN be the immersed part when the plate is in the position of equilibrium, then

$$ABMN = 4a^2s.$$

Let DEMN be the part immersed when the plate has been moved through an indefinitely small angle θ, in such a way that the quantity of fluid displaced after the disturbance is the same as that displaced before; and therefore

$$ACD = CEB.$$

Let G be the centre of gravity of the plate, g that of ABMN, and g' that of DEMN. Let the vertical line through g' meet the line gG in O, and then O is the metacentre.

If z be the distance of g' from the vertical line through G,

$$GO \sin\theta = z,$$

$$\text{or,}\quad GO = \frac{z}{\sin\theta} = \frac{z}{\theta},\ \text{since } \theta \text{ is indefinitely small.}$$

The value of z may be found as follows:—

$$DEMN = ABMN - ACD + CEB.$$

Therefore the moment of DEMN about the vertical line through G is equal to the algebraic sum of the moments of these several parts.

The area of DEMN is equal to $4a^2s$, and the distance of its centre of gravity from the vertical line through G is by hypothesis z.

The area of ABMN is equal to $4a^2s$, and the distance of its centre of gravity from the vertical line through G is equal to $gG \,.\, \theta$, or $a\theta\,(1 - s)$.

The area of the indefinitely small triangle ACD may be regarded as equal to that of a sector of the circle, whose centre is at C and radius CA or a; whence the area of ACD is equal to $\frac{1}{2}a^2\theta$, and the distance of its centre of gravity of ACD from C is equal to $\frac{2}{3}a$, and therefore its distance from the vertical line through G is $\frac{2}{3}a - m\mathrm{C}$.

Similarly, the area of CEB is equal to $\frac{1}{2}a^2\theta$, and the distance of its centre of gravity from the vertical line through G is equal to $\frac{2}{3}a + m\mathrm{C}$.

Therefore, assuming the left hand side of the vertical line through G to be the positive direction, and consequently giving the negative sign to the moment of CEB, we have

$$4a^2s \,.\, z = 4a^3s \,.\, \theta\,(1 - s) - \tfrac{1}{2}a^2\theta\,(\tfrac{2}{3}a - m\mathrm{C}) - \tfrac{1}{2}a^2\theta\,(\tfrac{2}{3}a + m\mathrm{C}),$$

and therefore,

$$z = a\theta \,.\, \frac{6s\,(1 - s) - 1}{6s}.$$

Consequently,

$$\mathrm{GO} = a \,.\, \frac{6s\,(1 - s) - 1}{6s}.$$

If this value be positive, O falls below G, and the equilibrium is unstable. If it equal zero, O coincides with G, or the equilibrium is neutral. If it be negative, O falls above G, or the equilibrium is stable. Hence, the equilibrium is

unstable if $s^2 - s + \frac{1}{6}$ be negative,
neutral ,, ,, vanish,
stable ,, ,, positive.

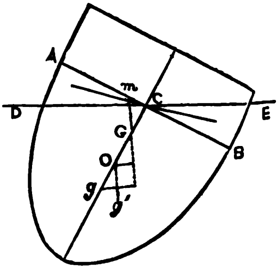
The roots of this expression are

$$\frac{3 + \sqrt{3}}{6} \text{ and } \frac{3 - \sqrt{3}}{6}.$$

Hence the equilibrium is unstable if the value of s lie between these limits, neutral if it be equal to either of them, and stable if it lie without them.

420. The method pursued in the preceding Article may be easily adapted to the more general case of a plate of any symmetrical figure, and also to that of any solid of revolution.

To find the metacentre of a plate of any symmetrical figure when floating with its axis vertical.



Let V be the magnitude of the part immersed. Let $2a$ be the line of floatation. Let A be the magnitude of the parts raised and depressed by the disturbance, which are evidently equal. Let p be the distance of the centre of gravity of either of these parts from C. Then, as in the preceding Article, taking the moments about the vertical line through G, we have

$$V\theta \,.\, GO = V\theta \,.\, gG - A(p - mC) - A(p + mC);$$

and therefore

$$GO = gG - \frac{2Ap}{V\theta}.$$

The parts raised and depressed may, as before, be regarded as small circular sectors, whose radius is a, and angle θ. Hence, $A = \frac{1}{2}a^2\theta$, and $p = \frac{2}{3}a$, consequently,

$$GO = gG - \frac{2}{3} \cdot \frac{a^3}{V}.$$

421. Ex. *An isosceles triangular plate floats vertically, with its base horizontal and its vertex immersed; to determine the position of the metacentre, and the nature of the equilibrium.*

Let 2α be the vertical angle, h the vertical height of the triangle, and s the specific gravity of the plate relatively to the fluid.

In the position of equilibrium, the part immersed is also an isosceles triangle, whose vertical angle is 2α. Let h' be its height, then

$$V = h'^2 \tan\alpha,$$

$$a = h' \tan\alpha,$$

and

$$gG = \frac{2}{3}(h - h').$$

Therefore,
$$GO = \tfrac{2}{3}(h - h') - \tfrac{2}{3}h' \tan^2 \alpha,$$
$$= \tfrac{2}{3}\left(h - \frac{h'}{\cos^2 \alpha}\right).$$

But,
$$h' = h\sqrt{s} \text{ (Art. 415); therefore}$$
$$GO = \frac{2h}{3}\left(1 - \frac{\sqrt{s}}{\cos^2 \alpha}\right).$$

Whence it follows that the equilibrium is unstable, neutral, or stable, according as s is less than, equal to, or greater than $\cos^4 \alpha$.

422. *To find the metacentre of any solid of revolution when floating with its axis vertical.*

Let V be the magnitude of the part immersed, and a the radius of the plane of floatation.

Let A be the magnitude of the parts raised and depressed by the disturbance, and p the distance of their centres of gravity from C; then, as in Art. 420,

$$GO = gG - \frac{2Ap}{V\theta}.$$

The parts raised and depressed may be regarded as parts of a sphere (radius $= a$,) cut out by planes intersecting at the centre at the indefinitely small angle θ; and therefore (Art. 385),

$$p = \frac{3\pi a}{16}.$$

Also,
$$A = \frac{2a^3\,\theta}{3}.$$

Substituting these values, we have

$$GO = gG - \tfrac{1}{4}\,.\,\frac{\pi a^4}{V}.$$

Cor. The second term on the right hand side of the equation is the distance of the metacentre from the centre of gravity of the displaced fluid, or

$$gO = \tfrac{1}{4}\,.\,\frac{\pi a^4}{V}.$$

423. Ex. 1. To find the metacentre of a right cone when floating with its axis vertical.

Let h be the height of the cone, $2a$ its vertical angle, and s its specific gravity relatively to the fluid. Then, if h' be the depth of the part immersed, the radius of the plane of floatation is equal to

$$h' \tan a,$$

and the volume of the part immersed is equal to

$$\tfrac{1}{3}\pi h'^3 \tan^2 a.$$

Also, the distance between the centres of gravity of the cone and of the part immersed is equal to

$$\tfrac{3}{4}(h - h').$$

Substituting these values in the general formula

$$GO = gG - \tfrac{1}{4}\frac{\pi a^4}{V},$$

we have for the metacentre of a right cone,

$$GO = \tfrac{3}{4}(h - h') - \tfrac{3}{4}h' \tan^2 a,$$

$$= \tfrac{3}{4}\left(h - \frac{h'}{\cos^2 a}\right).$$

But, by Art. 415, $h' = h\sqrt[3]{s}$; therefore,

$$GO = \frac{3h}{4}\left(1 - \frac{\sqrt[3]{s}}{\cos^2 a}\right).$$

Cor. Hence, the equilibrium of a right cone is stable when

$$\cos^2 a < \sqrt[3]{s};$$

$\therefore$ when $\cos^6 a < s.$

424. Ex. 2. To find the metacentre of a circular cylinder when floating with its axis vertical.

Let r be the radius of the cylinder, h its height, and s its specific gravity relatively to that of the fluid; then, if h' be the depth of the part immersed,

$$V = \pi r^2 h';$$

also, $a = r,$

and $gG = \tfrac{1}{2}(h - h').$

Substituting these values in the general expression of Art. 422, we have

$$GO = \tfrac{1}{2}(h-h') - \tfrac{1}{4}\frac{r^2}{h'}.$$

But $h' = hs$, and therefore

$$GO = \frac{2h^2s(1-s) - r^2}{4hs}.$$

Cor. Hence, the equilibrium is stable when

$$r^2 > 2h^2s(1-s);$$

that is, when

$$\frac{r^2}{h^2} > 2s(1-s).$$

425. Ex. To find the metacentre of a paraboloid of revolution.

Let $y^2 = 4cx$ be the equation of the generating parabola, h the axis of the paraboloid, and h' the depth of the part immersed; then

$$a^2 = 4ch',$$

and

$$V = 2\pi ch'^2.$$

Also (Art. 382),

$$gG = \tfrac{2}{3}(h-h');$$

therefore

$$GO = \tfrac{2}{3}(h-h') - 2c.$$

To find h', let w be the weight of a unit of the fluid; then, since the weight of the body is equal to that of the displaced fluid,

$$2\pi ch^2 . ws = 2\pi ch'^2 w;$$

whence

$$h' = h\sqrt{s}.$$

Consequently, for the metacentre of the paraboloid, we have

$$GO = \tfrac{2}{3}h(1-\sqrt{s}) - 2c.$$

MISCELLANEOUS EXAMPLES.

1. The plane surface of a spherical segment is the base of a cone; what must be the height of the cone that the solid may rest with any point of its spherical surface in contact with a horizontal plane, the axis of the segment being equal to half the radius of the sphere?

Let r be the radius of the sphere, then the height of the cone is equal to $$\frac{r(2+\sqrt{13})}{2}.$$

2. A cylindrical vessel, filled with liquid, is divided into four equal portions by two vertical planes through its axis at right angles to each other; find the resultant pressure upon either portion of the cylindrical surface.

Let a be the height of the vessel, r the radius of its base, and w the weight of a unit of the liquid, then the required pressure is equal to $$\frac{wa^2r\sqrt{2}}{2}.$$

3. Two balls of equal weight are fastened together by a cord of given length, one of the balls is placed on the ground, and the other projected vertically upwards with a given velocity, to what height will the second ball rise?

Let l be the length of the cord, and the velocity of projection that acquired by a body falling freely through the height h, then the required distance is equal to $\frac{h-l}{4}.$

4. Find, by Guldin's Properties, the surface of the spindle generated by the revolution of a circular arc about its chord.

Let r be the radius of the arc, and $2a$ the angle subtended by the arc at the centre of the circle, then the surface of the spindle is equal to $$4\pi r^2 (\sin a - a \, . \cos a).$$

5. A hemispherical bowl, filled with a fluid, is divided into two parts by a vertical plane through the centre, find the resultant pressure upon one of the parts.

Let r be the radius of the bowl, and w the weight of a unit of the fluid, then the pressure required is equal to

$$\tfrac{1}{3}wr^3\sqrt{(\pi^2+4)}.$$

6. A body is projected with a given velocity along a rough horizontal plane; find the space described by the body before coming to rest.

Let μ be the co-efficient of friction, and the velocity of projection that acquired in falling freely through the distance h, then the space required is equal to $\frac{h}{\mu}$.

7. A body is projected in a horizontal direction along a smooth inclined plane, with a velocity equal to that acquired in falling freely through the distance h; show that the path described is a parabola, and that if α be the inclination of the plane, the latus rectum of the parabola is equal to $\frac{4h}{\sin\alpha}$.

8. Find the velocity acquired by a body in falling down a rough inclined plane.

Let v_1 v_2 be the velocities acquired by a body in falling freely through distances equal to the height and base of the plane respectively, and let μ be the co-efficient of friction, then the required velocity is equal to $\sqrt{(v_1^2-\mu v_2^2)}$.

9. An imperfectly elastic ball is projected at a given inclination against a horizontal plane; determine the velocity of projection that the ball after its rebound may strike a given point in a vertical wall.

Let e be the modulus of elasticity, α the angle at which the ball strikes the horizontal plane, a the distance of the wall from the first point of impact, and b the height of the second point, then the velocity of projection is that acquired in falling freely through a distance equal to

$$\frac{a^2}{2ae \sin 2\alpha - 4b \cos^2\alpha}.$$

10. In a cylindrical vessel filled with air, a cone exactly fitting it is placed with its vertex downwards, to what depth will the cone sink, supposing it to come to rest before its vertex touches the bottom of the vessel?

Let a be the height of the cylinder, $3b$ that of the cone, s the specific gravity of the cone relatively to that of mercury, and h the height of the barometric column; then the required depth is equal to

$$\frac{bs(a-b)}{h+bs}.$$

11. A lever of equal arms, bearing equal weights P, P, is moveable about a rough cylindrical axis; find how far one of the weights may be moved towards the fulcrum without disturbing the equilibrium, disregarding the difference between the radius of the axis and that of the hole in which it works.

Let w be the weight of the lever, r the radius of the axis, and μ the co-efficient of friction, then, if $\tan \epsilon = \mu$, the distance required is equal to

$$\frac{r \sin \epsilon\,(2\mathrm{P}+w)}{\mathrm{P}}.$$

12. Determine the velocity with which a body must be projected up a rough inclined plane so that it may just reach the top, the inclination of the plane being equal to the angle of repose.

Let h be the height of the plane, then the required velocity is equal to

$$\sqrt{(4gh)}.$$

13. A thin conical vessel, when floating in a certain fluid with its vertex downwards, is immersed to half its depth; determine the nature of the equilibrium.

Let h be the height of the cone, then the equilibrium is stable if the radius of the base is greater than

$$\frac{h\sqrt{7}}{3}.$$

14. A body falls down an inclined plane which is partly smooth and partly rough, what must be the inclination of the plane, that the velocity acquired in falling down a given distance on the smooth plane shall be just sufficient to carry the body over an equal distance along the rough plane?

Let μ be the co-efficient of friction, then the inclination of the plane is equal to

$$\tan^{-1}\left(\frac{\mu}{2}\right).$$

15. Two planes of equal inclination meet in their lowest points, a perfectly elastic ball falling vertically strikes one of the planes at a given point, and, after rebounding from the second plane, rises vertically; from what height did the ball fall on to the first plane?

Let α be the inclination of the planes, and a the distance of the first point of impact from the lowest point of the plane, then the height required is equal to

$$\frac{a \cos \alpha}{\sin 4\alpha}.$$

16. A perfectly elastic ball falls into a hemispherical bowl from a given height; determine the position of the first point of impact, in order that the second may be the lowest point of the bowl.

Let h be the given height, and r the radius of the bowl, then, if θ be the angle subtended at the centre by the arc contained by the two points of impact,

$$\cos \theta = \frac{1}{4} + \frac{1}{4}\sqrt{\left(1 + \frac{r}{h}\right)}.$$

17. A circular plate just immersed in a liquid is divided by a horizontal diameter; find the centres of pressure of the upper and lower semicircles.

Let r be the radius of the circle, then the distance of the centre of pressure of the upper semicircle from the highest point of the plate is equal to

$$\frac{r}{4} \cdot \frac{15\pi - 32}{3\pi - 4},$$

and that of the lower semicircle is equal to

$$\frac{r}{4} \cdot \frac{15\pi + 32}{3\pi + 4}.$$

18. An elastic ball is projected with a given velocity from a point in a horizontal plane; at what distance from that point will it finally rest?

Let e be the modulus of elasticity, $\surd(2gh)$ the velocity, and a the angle of projection, then the distance required is equal to

$$\frac{2h \sin 2a}{1 - e}.$$

19. The axis of a thin hemispherical bowl, when on the point of slipping down an inclined plane, is inclined to the vertical line at an angle of 45°; determine the co-efficient of friction.

The co-efficient of friction is equal to

$$\frac{2\surd 2 + 1}{7}.$$

20. An elastic ball is projected from a point in one of two parallel vertical walls, and strikes against the other; compare the intervals between the times of the successive impacts.

Let t be the time between projection and the first impact, then the required intervals are

$$\frac{t}{e}, \frac{t}{e^2}, \frac{t}{e^3}, \text{ and so on.}$$

21. A hemispherical bowl, of given material, just floats in a given fluid; determine the thickness of the bowl.

Let s be the specific gravity of the material of the bowl relatively to the given fluid, then, if a be the external radius, the internal radius is equal to $$a\sqrt{\left(\frac{s-1}{s}\right)}.$$

22. A particle moves from rest over the convex side of a parabola whose axis is vertical; show that when the particle starts from the vertex, the square of the reaction at any point varies inversely as the cube of the distance from the focus.

23. A particle moves from rest over the convex side of a parabola, whose axis is horizontal and whose plane is vertical; show that when the particle starts from a point whose distance from the axis is equal to three times the semi latus rectum, it will leave the curve at the extremity of the latus rectum.

APPENDIX.

◆

I. *To find the length of a perpendicular* (p) *drawn to the tangent of an ellipse from any point* (S).

Let $a^2y^2 + b^2x^2 = a^2b^2$ be the equation to the ellipse, the centre being the origin of co-ordinates, then the equation to the tangent at the point (x', y') is

$$a^2yy' + b^2xx' = a^2b^2;$$

and ∴ if x, y be the co-ordinates of S,

$$p = \pm\frac{a^2yy' + b^2xx' - a^2b^2}{\sqrt{(a^4y'^2 + b^4x'^2)}};$$

$$= \pm\frac{a^2yy' + b^2xx' - a^2b^2}{ab\sqrt{(rr')}}.$$

Cor. 1. If S be at the centre, $x = 0$, and $y = 0$,

$$\therefore \qquad p = \frac{ab}{\sqrt{(rr')}}.$$

Cor. 2. If S be at the focus from which r is measured, $x = -ae$, and $y = 0$,

$$\therefore \qquad p = b\sqrt{\left(\frac{r}{r'}\right)}.$$

Cor. 3. If S be at the extremity of the major axis, or $x = -a$, and $y = 0$,

$$\therefore \qquad p = \frac{b(x'+a)}{\sqrt{(rr')}} = \frac{b \,.\, \mathrm{SM}}{\sqrt{(rr')}}.$$

Cor. 4. If S be at any point on the ellipse, then $a^2y'^2 + b^2x'^2 = a^2b^2$, and $a^2y^2 + b^2x^2 = a^2b^2$,

$$\therefore \qquad 2a^2b^2 = a^2(y^2 + y'^2) + b^2(x^2 + x'^2).$$

But
$$p = -\frac{2a^2yy' + 2b^2xx' - 2a^2b^2}{2ab\sqrt{(rr')}},$$

the negative sign being taken because S is on the same side of the tangent as the origin.

Substituting for $2a^2b^2$, the value given above, we have

$$p = \frac{a^2(y-y')^2 + b^2(x-x')^2}{2ab\sqrt{(rr')}},$$

II. To find the length of a perpendicular (p) drawn to the tangent of a parabola from any point S.

Let $y^2 = 4cx$ be the equation to the parabola, the vertex being the origin; then the equation to tangent at (x', y) is

$$yy' - 2cx = 2cx';$$

and therefore, if x, y be the co-ordinates of S,

$$p = \pm\frac{yy' - 2cx - 2cx'}{\sqrt{(4c^2 + y'^2)}},$$

$$= \pm\frac{yy' - 2cx - 2cx'}{2\sqrt{(cr)}}.$$

Cor. 1. If S be at the focus, $x = c$ and $y = 0$,

$$\therefore \qquad p = \sqrt{(cr)},$$

Cor. 2. If S be at the vertex, $x = 0$ and $y = 0$,

$$\therefore \qquad p = x'\sqrt{\left(\frac{c}{r}\right)}.$$

COR. 3. If S be any point on the parabola, then $y^2 = 4cx$, and

$$p = \frac{(y - y')^2}{4\sqrt{(cr)}}.$$

III. To find the length of perpendicular (p) drawn to the tangent of a hyperbola from any point (S).

Let $a^2y^2 - b^2x^2 = -a^2b^2$ be the equation to the hyperbola, then the equation to the tangent at (x', y') is

$$a^2yy' - b^2xx' = -a^2b^2,$$

and $\therefore$ if x, y be the co-ordinates of S,

$$p = \pm\frac{a^2yy' - b^2xx' + a^2b^2}{\sqrt{(a^4y'^2 + b^4x'^2)}}.$$

$$= \pm\frac{a^2yy' - b^2xx' + a^2b^2}{ab\sqrt{(rr')}}.$$

COR. 1. If S be at the centre, $x = 0$, and $y = 0$, and hence

$$p = \frac{ab}{\sqrt{(rr')}}.$$

COR. 2. If S be at the focus from which r is measured, $x = -ae$, and $y = 0$. Hence,

$$p = b\sqrt{\left(\frac{r}{r'}\right)}.$$

COR. 3. If S be any point in the hyperbola, then, proceeding as in I. Cor. 4, we have

$$p = \pm\frac{a^2(y - y')^2 - b^2(x - x')^2}{2ab\sqrt{(rr')}},$$

the upper sign if S and P are on the same branch, the lower sign if they are on different branches.

RADIUS OF CURVATURE.

Def. If P and Q be any two points in a curve, and the circle be drawn which passes through P and Q, and has its centre in the normal at P, then the limiting value of this circle as Q approaches more and more nearly to P, is called the *circle of curvature*, and its radius the *radius of curvature.*

Let ρ be the radius of curvature at any point P (fig. Art. 224), then ρ is the limiting value of PC, the radius of the circle PQq. But

$$\begin{aligned} PQ^2 &= 2PC \, . \, PX, \\ &= 2PC \, . \, QR, \end{aligned}$$

$$\therefore \qquad PC = \frac{PQ^2}{2QR},$$

$$\therefore \qquad \rho = \text{limit} \frac{PQ^2}{2QR}.$$

IV. *To find the radius of curvature at any point* (P) *of an ellipse.*

Let x, y be the co-ordinates of P, x', y' those of Q; then

$$PQ^2 = (x-x')^2 + (y-y')^2;$$

and since QR is the perpendicular from Q to the tangent at P, then (I. Cor. 4),

$$2QR = \frac{b^2(x-x')^2 + a^2(y-y'^2)}{ab\sqrt{(rr')}},$$

$$\therefore \quad \frac{PQ^2}{2 \, . \, QR} = ab\sqrt{(rr')} \, . \, \frac{(x-x')^2 + (y-y')^2}{b^2(x-x')^2 + a^2(y-y')^2},$$

$$= ab\sqrt{(rr')} \, . \, \frac{1 + \left(\dfrac{y-y'}{x-x'}\right)^2}{b^2 + a^2\left(\dfrac{y-y}{x-x'}\right)^2},$$

But the limit of $\frac{y-y'}{x-x'}$, as found in deducing the equation to the tangent, is $-\frac{b^2}{a^2}\frac{x}{y}$, $\therefore$

$$\rho = ab\surd(rr') \,.\, \frac{1+\dfrac{b^4x^2}{a^4y^2}}{b^2+\dfrac{a^2b^4x^2}{a^4y^2}},$$

$$= ab\surd(rr') \,.\, \frac{a^4y^2+b^4x^2}{a^2b^2(a^2y^2+b^2x^2)},$$

$$= ab\surd(rr') \,.\, \frac{a^2b^2rr'}{a^4b^4},$$

$$= \frac{(rr')^{\frac{3}{2}}}{ab}.$$

V. To find the radius of curvature at any point of a parabola.

As in the preceding,

$$PQ^2 = (x-x')^2 + (y-y')^2,$$

and by II. Cor. 3,

$$2QR = \frac{(y-y')^2}{2\surd(cr)};$$

$$\therefore \qquad \frac{PQ^2}{2QR} = 2\surd(cr) \,.\, \frac{(x-x')^2+(y-y')^2}{(y-y')^2};$$

$$= 2\surd(cr)\left\{\left(\frac{x-x'}{y-y'}\right)^2 + 1\right\}.$$

But the limit of $\frac{x-x'}{y-y'} = \frac{y}{2c}$; hence

$$\rho = 2\surd(cr)\left\{\left(\frac{y}{2c}\right)^2 + 1\right\},$$

$$= 2\surd(cr) \,.\, \frac{4cx+4c^2}{4c^2},$$

$$= 2\surd(cr) \,.\, \frac{c+x}{c},$$

$$= 2c \,.\, \left(\frac{r}{c}\right)^{\frac{3}{2}}.$$

VI. To find the radius of curvature at any point of a hyperbola.

As before, $$PQ^2 = (x - x')^2 + (y - y')^2,$$

and by III. Cor. 3, since P and Q are on the same branch,

$$2QR = -\frac{b^2(x - x')^2 - a^2(y - y')^2}{ab\sqrt{(rr')}},$$

$$\therefore \qquad \frac{PQ^2}{2QR} = -ab\sqrt{(rr')} \, . \, \frac{(x - x')^2 + (y - y')^2}{b^2(x - x')^2 - a^2(y - y')^2}.$$

$$= -ab\sqrt{(rr')} \, . \, \frac{1 + \left(\dfrac{y - y'}{x - x'}\right)^2}{b^2 - a^2\left(\dfrac{y - y'}{x - x'}\right)^2},$$

But limit of $\dfrac{y - y'}{x - x'} = \dfrac{b^2 x}{a^2 y}$,

$$\therefore \qquad \rho = -ab\sqrt{(rr')} \, . \, \frac{1 + \dfrac{b^4x^2}{a^4y^2}}{b^2 - \dfrac{a^2b^4x^2}{a^4y^2}},$$

$$= -ab\sqrt{(rr')} \, . \, \frac{a^4y^2 + b^4x^2}{a^2b^2(a^2y^2 - b^2x^2)},$$

$$= ab\sqrt{(rr')} \, . \, \frac{-a^2b^2(rr')}{-a^4b^4},$$

$$= \frac{(rr')^{\frac{3}{2}}}{ab}.$$

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